Lie Algebras
Exercises
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1. First problem set

Problem 1.1. Let A be an associative algebra over a field $\mathbb{F}$. Let $[\cdot,\cdot] : A \times A \to A$ be defined as $[x, y] = xy - yx$ for every $x, y \in A$ then $[\cdot,\cdot]$ is a Lie bracket on A.

Proof. Let $x, y, z \in A$ and $\alpha, \beta \in \mathbb{F}$ then $[\alpha x + \beta y, z] = (\alpha x + \beta y)z - z(\alpha x + \beta y) = \alpha(xz - zx) + \beta(yz - zy) = \alpha[x, z] + \beta[y, z]$.

Also $[z, \alpha x+\beta y] = z(\alpha x+\beta y)-(\alpha x+\beta y)z = (zx-xz)\alpha+(zy-yz)\beta = \alpha[z, x]+\beta[z, y]$. So, the bracket as defined is an F-bilinear map.

Now, consider $[x, x] = xx - xx = 0$ for ever $x \in A$.

Finally, we may proove the Jacobi identity, to this end, consider

$$[x, [y, z]]-[[x, y], z]+[y, [x, z]] = x(yz-zy)-(yz-zy)x-([x, y]z-z[x, y])+(y[x, z]-[x, z]y)$$

$$= x(yz - zy) - (yz - zy)x - (xy - yx)z + z(xy - yx) + y(xz - zx) - (xz - zx)y$$

$$= x(yz)-x(zy)-(yz)x-(zy)x-(xy)z-(yx)z+z(xy)-z(yx)+y(xz)-y(zx)-(xz)y-(zx)y = 0$$

This proves that the bracket defined is a Lie bracket. Notice that in the last step we need A to be associative, in order to get the desired cancellation. □

Problem 1.2. Let A be an associative algebra. Then the set of derivations of A, $Der(A) := \{\phi : A \to A \in gl(A) : \phi(xy) = x\phi(y) + \phi(x)y$ for every $x, y \in A\}$ defines a Lie subalgebra of $gl(A)$

Proof. To see that $Der(A)$ is a subspace let $\phi, \psi \in Der(A)$ then $(\phi + \psi)(xy) = \phi(xy) + \psi(xy) = x\phi(y) + \phi(x)y + x\psi(y) + \psi(x)y = x(\phi(y) + \psi(y)) + (\phi(x) + \psi(x))y = x(\phi+\psi)(y) + (\phi + \psi)(x)y$

Let $\alpha \in \mathbb{F}$ then $(\alpha\phi)(xy) = \phi((\alpha x)y) = \alpha x\phi(y) + \phi(\alpha x)y = \alpha(x\phi(y) + \phi(x)y) = \alpha\phi(x)$. This proves that $Der(A)$ is a subspace.

By definition of a subalgebra, we need that for any $\phi, \psi \in Der(A)$ then $[\phi, \psi] \in Der(A)$, i.e., $[\phi, \psi](xy) = x[\phi, \psi](y) + [\phi, \psi](x)y$. So, consider

$$[\phi, \psi](xy) = (\phi\psi - \psi\phi)(xy) = \phi(\psi(xy)) - \psi(\phi(xy)) = \phi(\psi(x)y + x\psi(y)) - \psi(\phi(x)y + x\phi(y))$$

$$= \phi\psi(x)y + \psi(x)\phi(y) + \phi(x)\psi(y) + x\phi\psi(y) - \psi\phi(x)y - \phi(x)\psi(y) - \psi(x)\phi(y) - x\psi\phi(y)$$

$$= \phi\psi(x)y + x\phi\psi(y) - \psi\phi(x)y - x\psi\phi(y) = (\phi\psi(x) - \psi\phi(x))y + x(\phi\psi(y) - \psi\phi(y))$$

$$= [\phi, \psi](x)y + x[\phi, \psi](y)$$

Which is the desired result. □

Problem 1.3. Let L be a Lie algebra. For $x \in L$ let $ad(x) \in gl(L)$ be defined by $ad(x)y = [x, y]$. Then, $ad(x) \in Der(A)$ (i.e. $ad(L) \subset Der(A)$ where $ad(L) = \{ad(x) : x \in L\}$)

Proof. We may prove that $ad(x) : L \to L$ is a Lie algebra's homomorphisim for a fixed $x \in A$. Since the bracket is bilinear then $ad(x)(\alpha y) = \alpha \cdot ad(x)y$ as well as $ad(x)(y + z) = ad(x)y + ad(x)z$. Consider $ad(x)(yz) = [x, yz] = x(yz) - (yz)x = (xy)z + (yx)z - y(xz) - y(zx) = ad(x)y \cdot z + y \cdot ad(x)z$. □

Problem 1.4. Let I and J be ideals in a Lie algebra L. Show that $I + J$, $I \cap J$, and $[I, J]$ are ideals in L.

Proof. First of all, lets write the definition of each subset,

- $I + J = \{x + y : x \in I, y \in J\}$
- $I \cap J = \{x : x \in I \text{ and } y \in J\}$
- $[I, J] = \text{span}\{[x, y] : x \in I \text{ and } y \in J\}$

So, we will prove that $I + J$ is an ideal. By definition, $I + J$ is closed under addition, so take $\alpha \in \mathbb{F}$ and $x + y \in I + J$ then $\alpha(x + y) = \alpha x + \alpha y \in I + J$ because $\alpha x \in I$ and $y \in J$. Finally we need $I + J$ to be closed under the bracket. Let $w \in L$ then $[x + y, w] = [x, w] + [y, w] \in I + J$ because $[x, w] \in I$ and $[y, w] \in J$.

$I \cap J$ is clearly a subspace, to see that it is an ideal, let $x, y \in I \cap J$. Then $[x, y] \in I$ and $[x, y] \in J$ because $x, y \in I$ and $x, y \in J$.

Now, let $\sum \lambda_i [x_i, y_i] \in [I, J]$ and $w \in L$ then $[\sum \lambda_i [x_i, y_i], w] = \sum \lambda_i [[x_i, y_i], w] = \sum \lambda_i ([x_i, [y_i, w]] - [[x_i, w], y_i])$ but $\in I$ and $[x_i, w] \in I$ and $[y_i, w] \in J$ so $[I, J]$ is an ideal.

□

Problem 1.5. Let I be an ideal of the Lie algebra L. Define $[\cdot, \cdot] : L/I \to L/I$ by $[x+I, y+I] = [x, y] + I$. Show that this bracket is a well defined Lie bracket in L/I.

Proof. Suppose that $x + I = x' + I \in L/I$ and $y + I = y' + I \in L/I$ then $[x, y] + I = [x + I, y + I] = [x' + I, y' + I] = [x', y'] + I$, so $[x, y] - [x', y'] \in I$ then $[x', y'] = [x, y] + i$ for some $i \in I$ then $[x', y'] \in [x, y] + I$.

First, notice that $[x + I, x + I] = [x, x] + I = I$ (I is the zero coset in L/I). Now

$$[x + I, [y + I, z + I]] + [z + I, [x + I, y + I]] + [y + I, [x + I, z + I]] =$$

$$[x+I, [y, z]+I]+[z+I, [x, y]+I]+[y+I, [x, z]+I] = [x, [y, z]]+[z, [x, y]]+[y, [x, z]]+I = I$$

This proves that the bracket is a well defined Lie bracket. □

Problem 1.6. Show that $ad : L \to gl(L)$ is a homomorphisim of Lie algebras, with kernel Z(L).

Proof. By bilinearity of the bracket we have $ad(\alpha x) = \alpha \cdot ad(x)$ and $ad(x+y) = ad(x)+ad(y)$. So, consider $x, y, z \in L$ then $ad([x,y])z = [[x,y],z] = [z,[y,x]] = [x,[y,z]] - [y,[x,z]] = ad(x)[y,z] - ad(y)[x,z] = ad(x)ad(y)(z) - ad(y)ad(x)(z) = [ad(x), ad(y)](z)$.

Now consider $x \in \ker(ad)$ iff $0 = ad(x)y = [x,y]$ for every $y \in L$, iff $x \in Z(L)$ therefore $\ker(ad) = Z(L)$. □

Problem 1.7. Show that the kernel of a homomorphism of Lie algebras is an ideal. Show that the image of a homomorphism is a subalgebra

Proof. Let $\phi \in gl(L)$, α a scalar in the field, and $x, y \in \ker\phi$. Then $\phi(x+y) = \phi(x)+\phi(y) = 0$ which implies that $x+y \in \ker\phi$. Also, $\phi(\alpha x) = \alpha\phi(x) = 0$, so we may conclude that $\ker\phi$ is a subspace of the Lie algebra. Let $x \in \ker\phi$ and $y \in L$ then $\phi([x,y]) = [\phi(x), \phi(y)] = [0, \phi(y)] = 0$ then $[x,y] \in \ker\phi$, hence $\ker\phi$ is an ideal.

Let $\phi \in gl(L)$, α a scalar in the field, and $x, y \in L$. So, $\phi(x) + \phi(y) = \phi(x+y)$ implies $\phi(x) + \phi(y) \in Im\phi$. Also, $\alpha\phi(x) = \phi(\alpha x)$, gives that $Im\phi$ is a subspace of the Lie algebra. Let $\phi(x), \phi(y) \in Im\phi$ then $[\phi(x), \phi(y)] = \phi([x,y])$, so $[\phi(x), \phi(y)] \in Im\phi$, hence $Im\phi$ is a subalgebra.

□

Problem 1.8. Let I be an ideal in a Lie algebra L. Show that the map $\pi : L \to L/I$ defined by $\pi(x) = x + I$ is a homomorphisim of Lie algebras.

Proof. First, we may note that π is a well defined map, since the cosets in L/I define a partition, then for every $x \in L$ there is a unique coset for which, $x \in x + I \subset L/I$. Now, consider $\pi(x+y) = x+y+I = (x+I)+(y+I) = \pi(x)+\pi(y)$ and $\pi(\alpha x) = \alpha x+I = \alpha(x+I) = \alpha\pi(x)$. Finally, we need to consider $\pi([x,y]) = [x,y]+I = [x+I, y+I] = [\pi(x), \pi(y)]$. This gives that π is a homomorphism of Lie algebras. □

2. Second problem set

Problem 2.1. Let $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Show that the basis $\{e, f, h\}$ of $sl(2, \mathbb{C})$ diagonalizes $ad(h)$. Find the eigenvalues.

Proof. Consider,

$$[h, e] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -2e$$

$$[h, f] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -2f$$

And obviously $[h, h] = 0$ then the basis diagonalizes $ad(h)$ □

Problem 2.2. Show that $sl(2,\mathbb{C})$ has no proper, nontrivial ideals.

Proof. Let $J \subset sl(2,\mathbb{C})$ be an ideal and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in J$, then $[e, A] = \begin{pmatrix} c & d-a \\ 0 & -c \end{pmatrix} \in J$, we now bracket this with h and find $[h, [e, A]] = 2\begin{pmatrix} 0 & a-d \\ 0 & c \end{pmatrix} \in J$. Finally, we hit this with e again, to get $-\frac{1}{2c}[e, [h, [e, A]]] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e \in J$ that means $e \in J$.

Take a look at $B = A - be = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in J$, because J is a subspace. Now, $\frac{1}{-2c}[h, B] = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = f \in J$. Then $h = [e, f] \in J$. Since $\{e, f, h\}$ are a basis then $J = sl(2,\mathbb{C})$, which yields the desired result. □

Problem 2.3. Fix a matrix S in $gl(n,\mathbb{F})$ and define $K_s = \{x \in gl(n,\mathbb{F}) : x^t S = -Sx\}$. Show that K_s is a Lie subalgebra of $gl(n,\mathbb{F})$.

Proof. Given $x, y \in K_s$ then $(x+y)^t S = (x^t + y^t)S = x^t S + y^t S = -Sx - Sy = -S(x+y)$ then $x + y \in K_s$. Given $\alpha \in \mathbb{F}$ then $(\alpha x)^t S = \alpha(x^t S) = \alpha(-Sx) = -S\alpha x$ so K_s is a subspace. Finally, consider $[x, y]^t S = (xy - yx)^t S = (xy)^t S - (yx)^t S = y^t(x^t S) - x^t(y^t S) = y^t(-Sx) - x^t(-Sy) = Sxy - Syx = S[x, y]$. Then $[x, y] \in K_s$, hence K_s is a subalgebra. □

Problem 2.4. Let V be a complex vector space of dimension n. Suppose $x \in gl(V)$ is diagonalizable with eigenvalues $d_1, ..., d_n$. Show that $ad(x)$ is diagonalizable with eigenvalues $d_i - d_j$, $1 \leq i, j \leq n$.

Problem 2.5. Let $I \subset L$ be an ideal. Show that L/I is abelian iff $L' \subset I$

Proof. $\Rightarrow$: If L/I is abelian then for every $x + I, y + I \in L/I$ we have $[x + I, y + I] = [x, y] + I = I$ then $[x, y] \in I$ for every $x, y \in L$ hence $L' \subset I$.

$\Leftarrow$: If $L' \subset I$ then for every $x, y \in L$, $[x, y] \in I$ so $[x+I, y+I] = [x, y]+I = I$ hence L/I is abelian.

□

Problem 2.6. Let I be an ideal in a Lie algebra L. Show that all ideals of L/I are of the form J/I where J is an ideal in L that contains I.

Proof. Let $I \subset L$ be an ideal. Consider $\mathcal{F} = \{J : I \subset J \text{ and J is an ideal in L}\}$ and $\mathcal{G} = \{K : \text{K is an ideal in L/I}\}$. Define $\Theta : \mathcal{F} \to \mathcal{G}$ by $\Theta(J) = J/I$.

We may show that Θ is a well-defined bijection, this yields the desired result. So first, we may note that Θ is well-defined, given J an ideal in L containing I, take $x + I \in J/I$

and $y+I \in L/I$ then $[x+I, y+I] = [x,y]+I = I$ since $x \in J$ and $y \in L$ and J is an ideal, then J/I is an ideal in L/I.

Now, $\Theta(I) = I$ then Θ is injective. To see that Θ is surjective, take an ideal $M \subset L/I$, let $x+I \in M$ and $y+I \in L/I$ then $[x+I, y+I] = [x,y]+I \in M$ then let $\tilde{M} = \Theta^{-1}(M)$, by the previous argument, $\tilde{M}$ is an ideal in L that contains I and $\Theta(\tilde{M}) = M$

Then Θ is a bijection between the set of ideals in L/I and the ideals in L containing I. □

Problem 2.7. Suppose that L is a two-dimensional Lie algebra over the field $\mathbb{F}$. Show that $L' = \mathbb{F}[x,y]$ where $\{x,y\}$ is any basis of L.

Proof. Since every element in L is a linear combination of $\{x,y\}$ then consider $m, m' \in L$ then $[m,m'] = [\alpha x + \beta y, \alpha' x + \beta' y] = [\alpha x, \beta' y] + [\beta y, \alpha' x] = (\alpha\beta' + \beta\alpha')[x,y]$, then this result can be extended linearly to $L' = \mathit{span}\,\{[w,z] : w, z \in L\} = \mathbb{F}[x,y]$. □

Problem 2.8. Let L be a Lie algebra. Show that $L/Z(L)$ is isomorphic to a subspace of gl(L).

Proof. Define $\pi : L \to gl(L)$ by $\pi(x) = ad(x)$ by Problem 1.6 π is a homomorphisim with kernel $Z(L)$ by Problem 1.7 $Im\pi$ is a subalgebra. Invoking the first isomorphism theorem completes the proof. □

Problem 2.9. Show that $gl(2,\mathbb{C}) \cong sl(2,\mathbb{C}) \oplus \mathbb{C}$

Proof. Let $\psi : \mathbb{C} \to sl(2,\mathbb{C})$ be an embeding defined as $\psi(z) = \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix}$. Notice that $\ker\psi = 0$, then $Im\psi \cong \mathbb{C}$. Now let $\phi : gl(2,\mathbb{C}) \to \mathbb{C}$ by $\phi(A) = Tr(A)$ then $\ker\phi = sl(2,\mathbb{C})$. By the first isomorphism theorem we have $gl(2,\mathbb{C})/sl(2,\mathbb{C}) \cong \mathbb{C}$. But $sl(2,\mathbb{C}) \oplus \mathbb{C}/sl(2,\mathbb{C}) \cong \mathbb{C}$, so, $gl(2,\mathbb{C}) \cong sl(2,\mathbb{C}) \oplus \mathbb{C}$. □

3. Third problem set

Problem 3.1. Show that L is solvable iff each $L^{(j)}$ is solvable. Verify that I is an ideal of L, then $(L/I)^{(j)} = (L^{(j)}+I)/I$. Show that L is solvable iff $ad(L)$ is solvable in $gl(L)$.

Proof. Suppose L is solvable then there exists $m \geq 1$ so that $L^{(m)} = 0$ since $L^{(m)} \subset L^{(n)}$ for every $n \leq m$, then every $L^{(n)}$ is solvable, since $\left(L^{(n)}\right)^{(m-n)} = 0$. On the other hand, if each $L^{(j)}$ is solvable, then there exists a $m \geq 1$ so that $L^{(m)} = 0$, hence L is solvable.

Now, notice that $x \in (L/I)'$ can be written as $x = \sum [x_j + I, y_j + I] = \sum [x_j, y_j] + I \in (L'+I)/I$ for $x_j, y_j \in L$, on the other hand, for $z \in (L'+I)/I$ and $w \in I$ we have $z = \sum [x_j, y_j] + w + I = \sum [x_j, y_j] + I = \sum [x_j + I, y_j + I] \in (L/I)'$. This gives $(L/I)' = (L'+I)/I$. Simple induction completes the proof.

Suppose that L is solvable, then $ad(L)$ is solvable, since it is the homomorphic image of a solvable Lie algebra. On the other hand, if $ad(L)$ is solvable, then there exists $m \geq 1$ so that $ad(L)^{(m)} = 0$ then note that $L^{(m)} \subset ad(L)^{(m)} = L^m = 0$ hence $L^{(m)} = 0$ thus L is solvable. □

Problem 3.2. Let L be a Lie algebra, I a solvable ideal, and K a solvable subalgebra. Does the argument given in class extend to prove that $I + K$ is a solvable subalgebra? Yes. If L is not solvable then $rad(L)$ is contained in every maximal solvable subalgebra.

Proof. Suppose L is not solvable, then $rad(L) \neq L$. Suppose K is a solvable subalgebra, which does not contain $rad(L)$ then $K + rad(L) \supset K$ is solvable, which contradicts maximality of K. □

Problem 3.3. Show that TFAE:

(1) L is solvable
(2) $ad_{[L,L]}(x)$ where $x \in [L, L]$ is a nilpotent endomorphism of $[L, L]$
(3) $[L, L]$ is nilpotent

Proof. (1) ⇒ (2): First of all notice that $ad_{[L,L]}(x)(z) \in [L^{(1)}, L^{(1)}]$ for all $z \in [L, L]$ and that $ad^n_{[L,L]}(x)(w) \in [L^{(1)}, L^{(n)}]$ for all $w \in [L, L]$. But L solvable implies that there exists $m \geq 1$ so that $L^{(m)} = 0$ hence $[L^{(1)}, L^{(m)}] = 0 = ad^m_{[L,L]}(x)$.
(2) ⇒ (3): Directly by applying Engel's theorem.
(3) ⇒ (1): If $[L, L]$ is nilpotent, then is solvable, then there exists $m \geq 1$ so that $(L')^{(m)} = L^{(m+1)} = 0$ then L is solvable. □

Problem 3.4. Show that the center of $gl(n, \mathbb{F})$ is the space of scalar multiples of the identity. Show that the center of $sl(n, \mathbb{F})$ is 0, unless $char(\mathbb{F})$ divides n, in which case the center is the space of scalar multiples of the identity.

Proof. Let $A = \alpha I$ and $B \in gl(n, \mathbb{F})$ then $[A, B] = (\alpha I)B - B(\alpha I) = \alpha B - \alpha B = 0$ then $A \in Z(gl(n, \mathbb{F}))$. On the other hand, let $C = (c_{ij}) \in Z(gl(n, \mathbb{F}))$. Then by computing $0 = [C, \sum_i E_{ij}] = C(\sum_i E_{ij}) - (\sum_i E_{ij})C$ for a fixed $1 \leq j \leq n$, we note that the only terms that cancel are the $c_i i$ so the rest must be zero. So far, C must be a diagonal matrix, but by comparing each term in the diagonal pairwise (by applying the bracket to the respective E_{ij}) we obtain that every element in the diagonal must be equal.

Consider $Z(sl(n, \mathbb{F})) \subset Z(gl(n, \mathbb{F}))$, then $A \in Z(sl(n, \mathbb{F}))$ implies $A = \alpha I$ then $Tr(A) = n\alpha = 0$, so either $\alpha = 0$ or $char(\mathbb{F}) = k \cdot n$ for some integer $k \geq 1$. □

Problem 3.5. Show that in $gl(n, \mathbb{F})$, the subalgebra of upper triangular matrices K is a maximal solvable subalgebra.

Proof. Since $K^{(n)} = 0$ then K is solvable. □

Problem 3.6. Show that in $gl(n, \mathbb{F})$, the transpose map $A \mapsto A^t$ satisfies $[A, B]^t = -[A^t, B^t]$. Show that a subalgebra K is solvable (resp. nilpotent) iff K^t is solvable (resp. nilpotent). Show that $rad(L)$ is invariant under the transpose.

Proof. $[A, B]^t = (AB - BA)^t = B^t A^t - A^t B^t = -[A^t, B^t]$. Suppose K is solvable (resp. nilpotent) then there exists $n \geq 1$ (resp $m \geq 1$) so that $0 = (K^{(m)})^t = [K^{(m-1)}, K^{(m-1)}]^t = -[(K^{(m-1)})^t, (K^{(m-1)})^t]$, (resp. $0 = (K^m)^t = [K, K^{m-1}]^t = -[K^t, (K^{m-1})^t]$). Then K^t is solvable (resp. nilpotent). By symmetry ($K = (K^t)^t$), we obtain the only if part. □

Problem 3.7. Show that the radical of $gl(n, \mathbb{F})$ is $Z(gl(n, \mathbb{F}))$. Identify the radical of $sl(n, \mathbb{F})$. Under what condition(s) will $sl(n, \mathbb{F})$ be semisimple?

Proof. Since $gl(n, \mathbb{F})$ is not solvable, then by problem (3.2) then $rad(gl(n, \mathbb{F})) \subset Z(gl(n, \mathbb{F}))$. On the other hand, since $Z(gl(n, \mathbb{F}))$ is a solvable ideal of $gl(n, \mathbb{F})$ then $Z(gl(n, \mathbb{F})) \subset rad(gl(n, \mathbb{F}))$.

For $sl(n, \mathbb{F})$, just as in (3.4) if $char(\mathbb{F})$ divides n, then $rad(sl(n, \mathbb{F}))$ is the space of multiples of the identity. Otherwise $sl(n, \mathbb{F})$ is semisimple. □

Problem 3.8. (1) Let $A : V \to V$ be a linear map ($dimV < \infty$). Suppose that $v \neq 0$ and for some power of A kills v. Let $r \geq 1$ be the least integer for which $A^r v = 0$. Let $W = span(B)$ where $B = \{A^{r-1}v, ..., Av, v\}$. Show that,

(a) W is A-invariant

(b) B is a linearly independent set. Find the matrix of $A_W : V \to V$ with respect to B.

(2) If $x : V \to V$ is linear $\alpha \in \mathbb{F}$, and a power of $A = x - \alpha I$ kills $v \neq 0$. Let $W = span\{A^j v : 0 \leq j\}$. Show that W is x-invariant and use part (1) to find a basis of W and the matrix of x with respect to this basis.

Proof. (1) Since $AB = \{A^{r-1}v, ..., Av\} \subset B$ then W is A-invariant. To see that B is linearly independent, consider the linear combination $\alpha_0 v + \alpha_1 Av + ... + \alpha_{r-1} A^{r-1} v = 0$, then apply A^{r-1} to obtain that $\alpha_0 A^{r-1} = 0$ hence $\alpha_0 = 0$, continuing inductively reducing by one the power of A to apply in each step, we obtain that $\alpha_j = 0$ for all $0 \leq j \leq r - 1$, hence B is linearly independent. By applying A to each term of the basis, we notice that the matrix will be consisting of ones in the supdiagonal and zeros everywhere else.

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

(2) Apply part (1) to $A = x - \alpha I$ to obtain the eigenspace of α.

$\square$

4. Fourth set of problems

Problem 4.1. Let $N : V \to V$ be a nilpotent linear map ($dimV < \infty$). Let $v \neq 0$. We call the space $span\left\{v, Nv, N^2v, ...\right\}$ the N-cyclic spaces generated by v. Show that V is the direct sum of N-cyclic subspaces.

Proof. By induction on $dimV$. If $dimV = 1$ then $V = span\{v\}$ for some $v \neq 0$ and there is nothing to prove.

Suppose that the conclusion of the proposition is true for any space with $dim < dimV$. Let $N : V \to V$ be a nilpotent linear map. Let $m \geq 1$ be the smallest integer so that $N^m \equiv 0$. Let $K = \ker N := \{x \in V : Nx = 0\}$. If $\ker N = \{0\}$ then N cannot be nilpotent, so $dim(\ker N) \geq 1$.

Consider $\tilde{V} = V/\ker N$. Define $\tilde{N} : \tilde{V} \to \tilde{V}$ by $\tilde{N}(v + \ker N) = N(v) + \ker N$. Clearly if N is nilpotent, so is $\tilde{N}$. Since $dim\tilde{V} < dimV$ then, $\tilde{V}$ is the product of N-cyclic spaces, say

$$\tilde{V} = \bigoplus_{1 \leq j \leq r} \left\{\bar{v}_j, \tilde{N}\bar{v}_j, \tilde{N}^2\bar{v}_j, ..., \tilde{N}^{m_j}\bar{v}_j\right\}$$

by induction hypothesis, and they are all linearly independent. Notice that $\tilde{N}^{m_j+1}\bar{v}_j = 0$ which implies that $N^{m_j+1}v_j \in \ker N$ now we will claim that $\left\{N^{m_j+1}v_j : 1 \leq j \leq r\right\}$ are linearly independent. Consider a linear combination

$$0 = \sum_{j=1}^{r} a_j N^{m_j+1} v_j = N\left(\sum_{j=1}^{r} a_j N^{m_j} v_j\right)$$

then, $a_j N^{m_j} v_j \in \ker N$ so $\tilde{N}\left(\sum_{j=1}^{r} a_j \tilde{N}^{m_j}\bar{v}_j\right) = 0 = \sum_{j=1}^{r} a_j \tilde{N}^{m_j+1}\bar{v}_j$, but then $a_j = 0$ for every $1 \leq j \leq r$. But then the $\left\{v_j, Nv_j, ..., N^{m_j+1}v_j\right\}$ generate V, as desired. $\square$

Problem 4.2. (1) Show that if $x : V \to V$ is diagonalizable and the subspace W is x-invariant, then the restriction of x to W is diagonalizable.
(2) Show that if $x, y : V \to V$ are diagonalizable and $[x, y] = 0$, then x and y are simultaneously diagonalizable. (i.e. there exists an eigenbasis of B for both x and y).

Proof. (1) Let $V = V_{\lambda_1} \oplus ... \oplus V_{\lambda_n}$ where $\{\lambda_1, ..., \lambda_n\}$ are eigenvalues of x then by the modulo law, since W is invariant under x, we have $W = (W \cap V_{\lambda_1}) \oplus ... \oplus (W \cap V_{\lambda_n})$, which gives that W is the direct sum of eigenspaces of the restriction of x, hence the restriction of x is diagonalizable.

(2) Let $V = V_{\lambda_1} \oplus ... \oplus V_{\lambda_n}$ where $\{\lambda_1, ..., \lambda_n\}$ are eigenvalues of x then let $v \in V_{\lambda_i}$, since $[x, y] = 0$ then $xy(v) = yx(v) = y(\lambda_i v) = \lambda_i y(v)$ but then y leaves V_{λ_i} invariant, so by (1) y restricted to each V_{λ_i} is diagonalizable. Then, there exists a basis of eigenvectors of y that form V_{λ_i} wich are also eigenvectors of x. So there exists a basis of common eigenvectors for x and y, so they are simultaneously diagonalizable.

□

Problem 4.3. If $\mathbb{F}$ is an algebraically closed field, then there exist x-invariant subspaces $V_i \subset V$ with $V = V_1 \oplus ... \oplus V_m$ so that $x|_{V_i}$ has a matrix with respect to some basis with the form

$$x|_{V_i} \mapsto \begin{pmatrix} a_1 & 1 & 0 & ... & 0 \\ 0 & a_2 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & ... & a_m \end{pmatrix}$$

Proof. By the primary decomposition theorem we have that $V = \bigoplus_{j=1}^{r} \ker(x - a_j I_v)^{m_j} = \bigoplus_{j=1}^{r} W_j$, where each W_j is x-invariant.

Now consider $N_j = (x - a_j I_v)|_{W_j} : W_j \to W_j$. Since $W_j = \ker(x - a_j I_v)^{m_j}$ then $N_j^{m_j} \equiv 0$ so N_j is nilpotent, then by Problem (4.1) each W_j decomposes into N_j-cyclic subspaces, then Problem (3.8) gives the form of the matrix.

□

Problem 4.4. Show that for $x \in gl(V)$ the Jordan decomposition $x = x_s + x_n$ into semisimple and nilpotent parts is unique.

Proof. Suppose there are two Jordan decompositions $x = x_s + x_n = x'_s + x'_n$ then $x_n - x'_n = x'_s - x_s$ is a both nilpotent and semisimple operator, hence $x_n - x'_n = x'_s - x_s = 0$ so $x_s = x'_s$ and $x_n = x'_n$. □

Problem 4.5. Show that the Lie algebra L is semisimple iff L contains no abelian ideals.

Proof. Since every abelian ideal is automatically solvable then L semisimple cannot contain any abelian ideal, by the definition of semisimplicity.

On the other hand, suppose that L contains no abelian ideals and suppose I is a solvable ideal of L, then there exists $m \geq 1$ so that $I^{(m)} = 0$ but then $I^{(m-1)}$ is abelian, which is a contradiction, so L contains no solvable ideals, hence L is semisimple. □

Problem 4.6. Show that the Lie algebra L is solvable iff $[L, L] \subset rad(\kappa)$ where κ is the Killing form. (i.e. $\kappa(x, y) = Tr(ad(x), ad(y))$, for all $x, y \in L$).

Proof. $\Leftarrow$ Suppose that $[L, L] \subset rad(\kappa)$ then $\kappa(x, y) = 0$ for all $x \in [L, L]$ and $y \in L$ then by the corollary of Cartan's criterion, L is solvable.

⇒ If L is solvable, then by Lie's theorem we have that $x \in L$ is upper triangularized, in particular $Tr(ad(x), ad(y)) = 0$ for any $x, y \in L$ so $[L, L] \subset rad(\kappa)$. □

Problem 4.7. Let $B : V \times V \to \mathbb{F}$ be a symmetric bilinear form on V $(dimV < \infty)$. fix a basis $\beta = \{v_1, ..., v_n\}$ of V and define the $n \times n$ matrix $M = [B(v_i, v_j)]$. Let $v \in V$. Note that if $a_i \in \mathbb{F}$ satisfy the equation $v = \sum_{i=1}^{n} a_i v_i$, then the column vector $[a_1, ..., a_n]^t$ is a solution of the matrix equation

(1) $$M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} B(v, v_1) \\ \vdots \\ B(v, v_n) \end{pmatrix}$$

(1) Show that B is nondegenerate iff $det(M) \neq 0$
(2) Assume B is nondegenerate and suppose W is a subspace of V. Let $W^\perp = \{v \in V : B(v, w) = 0 \text{ for all } w \in W\}$. Show that $dimV = dimW + dimW^\perp$.
(3) Give a low-dimensional example of a vector space V, a nondegenerate symmetric bilinear form B and a subspace W for which $W \cap W^\perp \neq 0$.
(4) Assume B is nondegenerate and suppose W is a subspace of V for which $W \cap W^\perp = 0$. Show that $V = W \oplus W^\perp$, $W = (W^\perp)^\perp$, and $B|_W$ is non-degenerate.

Proof. (1) If B is nondegenerate then the only solution of the homogeneous system (1) has to be the trivial solution, so $det(M) \neq 0$. The same for the converse.
(2) Let $\tilde{w} \in W^\perp$ and write it as $\tilde{w} = \sum a_i v_i$ then,

$$M \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} B(\tilde{w}, v_1) \\ \vdots \\ B(\tilde{w}, v_n) \end{pmatrix}$$

then $B(\tilde{w}, v_i) = \sum a_j B(v_j, v_i)$ then for every v_i in the basis of W, $B(\tilde{w}, v_i) = 0$, but by nondegeneracy, we must have that $B(v_i, v_j) = 0$ if $v_i \in W$ and $v_j \in W^\perp$ then $dimW^\perp + dimW \leq dimV$ but $V = W \cup W^{bot}$ so $dimW^\perp + dimW = dimV$.
(3) The fact that $V = W \oplus W^\perp$ follows from the previous item and nondegeneracy gives that $W \cap W^\perp = 0$. Now let $w \in W^\perp$ then $B(v, w) = 0$ for every $v \in (W^\perp)\perp$ so $(W^\perp)\perp \subset W$, on the other hand if $w \in W$ then $B(v, w) = 0$ for every $v \in W^\perp$ then $W \subset (W^\perp)\perp$.

□

5. Fifth set of problems

Problem 5.1. Let L_i, $1 \leq i \leq n$ be Lie algebras and let $L = L_1 \oplus ... \oplus L_n$ denote de exterior direct sum.

(1) Show that $\tilde{L}_i = (0, 0, ..., L_i, 0, ...0)$ is an ideal of L

(2) Show that $Z(L) = Z(L_1) \oplus ... \oplus Z(L_n)$ and $[L, L] = [L_1, L_1] \oplus ... \oplus [L_n, L_n]$.

Proof. Let $x \in \tilde{L}_i$ and $y \in L$ be written as $x = (0, ..., 0, x_i, 0, ...0)$ and $y = (y_1, ..., y_n)$ then $[x, y] = \sum[x_i, y_i] = [0, y_1] + ... + [x_i, y_i] + ... + [0, y_n] = (0, 0, ..., [x_i, y_i], 0, ..., 0) \in \tilde{L}_i$, then $\tilde{L}_i$ is an ideal.

The other part follows from theorem in class and induction. □

Problem 5.2. Suppose that L is a Lie algebra, I_i are ideals of L, and that $L = I_1 \oplus ... \oplus I_n$ as vector spaces. Then $L \cong I_1 \oplus ... \oplus I_n$ (exterior direct sum).

Proof. Let $\tilde{L} = I_1 \oplus ... \oplus I_n$ as exterior direct sum, we may show that $L \cong \tilde{L}$. Define $\phi : \tilde{L} \to L$ for $x = x_1 + ... + x_n \in \tilde{L}$ as $\phi(x) = (x_1, ..., x_n)$. Now the representation of $x = x_1 + ... + x_n$ is unique, then ϕ is well-defined. Also $\phi(x) = 0$ iff $x = 0$ is clear, so $\ker \phi = \{0\}$ hence ϕ is injective. Let $(x_1, ..., x_n) \in L$ then $\phi(x) = \phi(x_1 + ... + x_n) = (x_1, ..., x_n)$ and ϕ is surjective.

Now if $\alpha, \beta \in \mathbb{F}$ and $x = x_1 + ... + x_n$ and $y = y_1 + ... + y_n$ then $\phi(\alpha x + \beta y) = (\alpha x_1 + \beta y_1, ..., \alpha x_n + \beta y_n) = \alpha(x_1, ..., x_n) + \beta(y_1, ..., y_n) = \alpha\phi(x) + \beta\phi(y)$. So, ϕ is a linear map.

Now,

$$\phi([x, y]) = \phi([x_1, y_1] + ... + [x_n, y_n]) = ([x_1, y_1], ..., [x_n, y_n])$$

$$= [(x_1, ..., x_n), (y_1, ..., y_n)] = [\phi(x), \phi(y)]$$

and ϕ is a Lie algebras isomorphism and this gives the desired result. □

Problem 5.3. (1) If L is simple then L is semisimple, $L' = L$, and $Z(L) = 0$.

(2) Suppose $L = I_1 \oplus ... \oplus I_n$ (as vector spaces) where each I_i is a simple ideal in L. Let $p_i : L \to I_i$ be defined by $p_i(x) = x_i$ where $x = (x_1, ..., x_n)$. Show that p_i is a Lie algebras homomorphism.

(3) Suppose $J \subset L$ is an ideal then $p_i(J) \subset I_i$ is an ideal of I_i.

(4) L is semisimple.

Proof. (1) Since $L' \subset L$ is an ideal, then either $L' = \{0\}$ or $L' = L$ but if $L' = \{0\}$ then L is abelian, but by definition of simple Lie algebra, L is not abelian, so $L' = L$, which implies that L is semisimple. Since $L^{(n)} = L$ for any $n \geq 1$. The same for $Z(L) = L$ implies L is abelian, so it follows that $Z(L) = \{0\}$.

(2) Let $\alpha, \beta \in \mathbb{F}$. Let $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$. Then $p_i(\alpha x + \beta y) = p_i((\alpha x_1 + \beta y_1, ..., \alpha x_n + \beta y_n)) = \alpha x_i + \beta y_i = \alpha p_i(x) + \beta p_i(y)$ and $p_i([x, y]) = p_i(([x_1, y_1], ..., [x_n, y_n])) = [x_i, y_i] = [p_i(x), p_i(y)]$. Then p_i is a Lie algebras homomorphism.

(3) Let $x = (x_1, ..., x_n) \in J$ and $y = (y_1, ..., y_n) \in L$ then $[x, y] = ([x_1, y_1], ..., [x_n, y_n]) \in J$, then $p_i([x, y]) = [x_i, y_i] = [p_i(x), p_i(y)] \in p_i(J)$ this is true for any arbitrary $y \in L$

and $x \in J$. So, for any $x' \in p_i(J)$ and $y' \in I_i$, then $[x', y'] \in p_i(J)$, so $p_i(J)$ is an ideal in I_i.

(4) Consider $L' \subset L$ be an ideal, by the previous result $p_i(L')$ is an ideal of I_i for each $1 \le i \le n$ but each I_i is simple, so either $p_i(L') = I_i$ or $p_i(L') = 0$, if $p_i(L') = I_i$ for all $1 \le i \le n$ then $L' = L$ and L is semisimple. So, suppose this is not true for some $1 \le j \le n$ then $p_i(L') = 0$ but this implies that $I_i' = 0$ so I_i is abelian, but I_i is simple so this cannot happend. So, L is semisimple.

□

Problem 5.4. Suppose that $0 \subset I_1 \subset I_2 \subset ... \subset I_n = L$ where $dim I_k = k$ then L is solvable.

Proof. Since $dim I_{i+1}/I_i = 1$ then each quotient is abelian. So, we may apply theorem in class to conclude that L is solvable. □

Problem 5.5. Suppose that $\phi : L \to gl(V)$ is a representation of L. Let $\tilde{L} = L/\ker\phi$.

(1) Show that $\tilde{\phi} : \tilde{L} \to gl(V)$ defined by $\tilde{\phi}(x + \ker\phi) = \phi(x)$ is a well-defined faithful representation.

(2) ϕ is irreducible iff $\tilde{\phi}$ is irreducible.

Proof. (1) To see that $\tilde{\phi}$ is well-defined, let $x + \ker\phi = x' + \ker\phi$ then $\phi(x) = \tilde{\phi}(x + \ker\phi) = \tilde{\phi}(x' + \ker\phi) = \phi(x')$, so $\tilde{\phi}$ is well-defined. Also $\tilde{\phi}(x + \ker\phi + y + \ker\phi) = \tilde{\phi}(x + y + \ker\phi) = \phi(x + y) = \phi(x) + \phi(y) = \tilde{\phi}(x + \ker\phi) + \tilde{\phi}(y + \ker\phi)$ and $\tilde{\phi}(\alpha x + \ker\phi) = \phi(\alpha x) = \alpha\phi(x) = \alpha\tilde{\phi}(x + \ker\phi)$. So $\tilde{\phi}$ is a well-defined linear map. Now, $\tilde{\phi}([x + \ker\phi, y + \ker\phi]) = \tilde{\phi}([x, y] + \ker\phi) = \phi([x, y]) = [\phi(x), \phi(y)] = [\tilde{\phi}(x), \tilde{\phi}(y)]$ gives that $\tilde{\phi}$ is a representation.

To show that $\tilde{\phi}$ is faithful, let $x + \ker\phi \in \ker\tilde{\phi}$ then $\tilde{\phi}(x + \ker\phi) = \phi(x) = 0$ so $x \in \ker\phi$ so $x + \ker\phi = 0$ in $\tilde{L}$ and so $\tilde{\phi}$ is faithful.

□

Problem 5.6. (1) Show that if the L-module V decomposes as $V = U \oplus W$ where U and W are invariant. Then $p_u(U + W) = U$ lies in $Hom_L(V, U)$.

(2) Suppose that the L-module V is completely reducible and that $Hom_L(V, V) = \mathbb{F}I_v$ then V is irreducible.

Proof. (1) Let $u + w$ and $u' + w' \in V$. Let $\alpha, \beta \in \mathbb{F}$ then $p_u(\alpha(u + w) + \beta(u' + w')) = p_u(\alpha u + \beta u' + \alpha w + \beta w') = \alpha u + \beta u' = \alpha p_u(u + w) + \beta p_u(u' + w')$. Now let $x \in V$ then U, W invariant, implies $xu \in U$ and $xw \in U$ so $p_u(x(u + w)) = p_u(xu + xw) = xu = xp_u(u + w)$ then p_u is an intertwinning map.

(2) Since V is completely reducible then every submodule has a complement, but then the projection map lies in $Hom_L(V, V)$ then $p_u(U+W) = U$ implies that $W = \{0\}$, then V is irreducible.

□

6. Sixth Problem Set

Problem 6.1. Suppose that V and W are L-modules. Show that the following spaces are L-modules with the given definitions:

(1) For $f \in V^*$ and $x \in L$ define $x.f$ by $(x.f)(v) = -f(x.v)$.
(2) For $v \in V$, $w \in W$ and $x \in L$, define $x.(v \otimes w) = (x.v) \otimes w + v \otimes (x.w)$
(3) For $f \in Hom_{\mathbb{F}}(V, W)$, $v \in V$ and $x \in L$ define $x.f \in Hom_{\mathbb{F}}(V, W)$ by $(x.f)(v) = x.f(v) - f(x.v)$. Show that the definition follows from the two previous ones and the isomorphism $Hom_{\mathbb{F}}(V, W) \cong V^* \otimes W$.

Proof. (1) Let $x, y \in L$, and $f, g \in V^*$ then $x.(f+g)(v) = -(f+g)(x.v) = -f(x.v) - g(x.v) = (x.f)(v) + (x.g)(v)$ also $((x+y).f)(v) = -f((x+y).v) = -f(x.v+y.v) = -f(x.v) - f(y.v) = (x.f)(v) + (y.f)(v)$ and $([x, y].f)(v) = -f([x, y].v) = -f(xy.v - yx.v) = -f(xy.v) + f(yx.v) = (xy.f)(v) - (yx.f)(v)$ so V^* is an L-module.

(2) Let $x, y \in L$, $v, v' \in V$ and $w, w' \in W$ then $(x+y).(v \otimes w) = ((x+y).v) \otimes w + v \otimes ((x+y).w) = (x.v+y.v) \otimes w + v \otimes (x.w+y.w) = (x.v+y.v) \otimes w + v \otimes (x.w+y.w) = x.v \otimes w + y.v \otimes w + v \otimes x.w + v \otimes y.w = x.(v \otimes w) + y.(v \otimes w)$

Also $x.(v \otimes w + v' \otimes w') = x.((v+v') \otimes (w+w')) = (x.(v+v')) \otimes (w+w') + (v+v') \otimes (x.(w+w')) = x.v \otimes w + x.v' \otimes w' + v \otimes x.w + v' \otimes x.w' = x.(v \times w) + x.(v' \times w')$. Now $[x, y].(v \times w) = (xy - yx).(v \times w) = ((xy - yx).v) \otimes w + v \otimes ((xy - yx).w) = (xy.v) \otimes w - yx.v \otimes w + v \otimes (xy.w) - v \otimes (yx.w) = xy.v \otimes w + v \otimes (xy.w) - (yx.v \otimes w + v \otimes (yx.w)) = xy.(v \otimes w) - yx.(v \otimes w)$.

(3) Given $f \in Hom_{\mathbb{F}}(V, W)$ there exists $g \in V^*$ and $w \in W$ so that $f \mapsto (g, w)$ then $(x.f)(v) = x.(g \otimes w)(v) = (x.g \otimes w)(v) + (g \otimes x.w)(v) = -g(x.v) \otimes w + g(v) \otimes x.w = x.(f(v)) - f(x.v)$.

□

Problem 6.2. Fix a bilinear, nondegenerate, symmetric form B on a Lie algebra L. Suppose that ϕ is a representation of L. We defined the Casimir c_ϕ in terms of the basis of L (and its dual basis). Show that the expression for c_ϕ is independent of the basis.

Proof. Consider two basis for L, $\mathcal{B} = \{x_1, ..., x_n\}$ and $\mathcal{B}' = \{x'_1, ..., x'_n\}$, and their corresponding dual basis with respect to B, $\mathcal{A} = \{y_1, ..., y_n\}$ and $\mathcal{A}' = \{y'_1, ..., y'_n\}$. Then we can write $x_i = \sum_j a_{ij} x'_j$ and in this fashion get the matrix $A = (a_{ij})$ so that $(x_1, ..., x_n)^t = A(x'_1, ..., x'_n)^t$.

Let $X = (x_1, ..., x_n)^t$ and $Y = (y_1, ..., y_n)^t$ as well as the primed vectors. If we denote by M the matrix associated to the bilinear form, then $I_n = XMY$ and $I_n = X'MY' = A^{-1}AX'MY' = A^{-1}XMY'$ but then $y_i = \sum_j a^{ji} y'_j$ where $A^{-1} = (a^{ij})$. So now, we may consider,

$$c_\phi(B) = \sum_{i=1}^{n} \phi(x_i)\phi(y_i) = \sum_{i=1}^{n} \phi\left(\sum_{j=1}^{n} a_{ij} x'_j\right) \phi\left(\sum_{j=1}^{n} a^{ji} y'_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} a^{ji} \phi\left(x'_j\right)\phi\left(y'_j\right)$$
$$= \sum_{i=1}^{n} \phi\left(x'_j\right)\phi\left(y'_j\right)$$

Which is the desired result. □

Problem 6.3. Let L be a semisimple Lie algebra and suppose that $\phi : L \to gl(V)$ is a faithful repersentation (where $\mathbb{F}$ is an algebraically closed field with characteristic zero and $dimV < \infty$)

(1) If V is irreducible and $B_\phi(x, y) = Tr(\phi(x)\phi(y))$ then $c_\phi = \frac{dimL}{dimV} I_V$.
(2) If ϕ is not faithful, show that there exists an ideal $I \subset L$ so that $\phi_I : I \to gl(V)$ is faithful and that $c_{\phi_I} \in Hom_L(V, V)$.

Proof. (1) Notice that $Tr(c_\phi) = \sum B_\phi(x_i, y_i) = dimL$. Since ϕ is faithful and L semisimple then V is completely reducible and ϕ acts by $\phi|_W = \lambda id_W$ then $c_\phi = \alpha I_V$ so $Tr(c_\phi) = \alpha dimV = dimL$ so $c_\phi = \frac{dimL}{dimV} I_V$
(2) Suppose that ϕ is not faithful, then $ker\phi \neq 0$. Since L semisimple then $ker\phi = \bigoplus_{j \in S} I_j$ then let $I = \oplus_{j \in S^c} I_j$ then ϕ_I is a faithful representation on I.

□

Problem 6.4. We say that a Lie algebra L is reductive if $rad(L) = Z(L)$.

(1) If L is reductive, show that L is completely reducible as an $ad(L)$-module.
(2) Prove the converse of (1)
(3) Show that if L is reductive, then $L = [L, L] \oplus Z(L)$, where $[L, L]$ is semisimple.
(4) Prove the converse of (3)

Proof. (1) Let $\phi : ad(L) \to gl(L)$ by $\phi(ad(x)) = ad(x)$, clearly ϕ is one-to-one. If $ad(L)$ is solvable then $L' \subset L$ and L/L' is abelian. Also $L/rad(L) = L/Z(L)$ has no proper ideals, so either $rad(L) = 0$ or $rad(L) = L$ if $rad(L) = L$ then L would be solvable then $ad(L) = 0$ hence completely reducible. Otherwhise if $rad(L) = 0$ then L is semisimple then L is completely reducible.
(2) If $L = I_1 \oplus ... \oplus I_n$ where each I_j is an ideal then $[L, L] = [I_1, I_1] \oplus ... \oplus [I_n, I_n]$. Since each ideal is simple, then either $[I_j, I_j] = I_j$ or $[I_j, I_j] = 0$ in the second case $I_j \subset rad(L)$ and $I_j \subset Z(L)$ hence $rad(L) = Z(L)$.

(3) If L is reductive, then $\ker ad = Z(L) = rad(L)$ and $\Im ad = L'$ gives $L = rad(L) \oplus L'$, now L' semisimple comes from the fact that $L' \cong L/Z(L) = L/rad(L)$ is semisimple.

(4) If $L = [L, L] \oplus Z(L)$, where $[L, L] \cong L/Z(L)$ is semisimple, then by correspondence theorem there is no solvable ideal that contains $Z(L)$ hence $rad(L) = Z(L)$, hence L is reductive.

□

7. Seventh problem set

Problem 7.1. Let $L = sl(n+1, \mathbb{F})$. Let $H = span\{E_{ii} - E_{i+1,i+1} : 1 \le i \le n\}$. Show that the root vectors are E_{ij} $(i \ne j)$.

Proof. Let $h = \sum \lambda_i (E_{ii} - E_{i+1,i+1}) \in H$ then $[h, E_{ij}] = \sum \lambda_i (E_{kk}E_{ij} - E_{k+1,k+1}E_{ij} - E_{ij}E_{kk} + E_{ij}E_{k+1,k+1}) = \sum \lambda_i (\delta_{ki}E_{ij} - \delta_{k+1,i}E_{ij} - \delta_{kj}E_{ij} + \delta_{j,k+1}E_{ij}) = \lambda_i E_{ij} - \lambda_{i-1}E_{ij} - \lambda_j E_{ij} + \lambda_{j-1}E_{ij} = (\lambda_i - \lambda_{i-1} - \lambda_j + \lambda_{j-1})E_{ij}$ □

Problem 7.2. Let $L = sp(2n, \mathbb{F}) = \left\{ x \in gl(2n, \mathbb{F}) : \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} x = -x^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}$

(1) Write $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ find all the conditions on x so that $x \in L$

(2) Let $H = span\{E_{ii} - E_{i+n,i+n} : 1 \le i \le n\}$. Show that the root vectors are: $\{E_{ij} - E_{j+n,i+n} : 1 \le i \ne j \le n\} \cup \{E_{i,i+n} : 1 \le i \le n\} \cup \{E_{i,j+n} + E_{j,i+n}\}$

Proof. When block multiplying

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^t & c^t \\ b^t & d^t \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

one obtains the following conditions: $-a = d^t$, $b = b^t$, and $c = c^t$.

Let $h = \sum \lambda_i (E_{ii} - E_{i+n,i+n}) \in H$ then

$$\begin{aligned}
[h, E_{ij}] - [h, E_{j+n,i+n}] &= \sum \lambda_k (E_{kk}E_{ij} - E_{k+n,k+n}E_{ij} - E_{ij}E_{kk} + E_{ij}E_{k+n,k+n}) \\
&- \sum \lambda_k (E_{kk}E_{j+n,i+n} - E_{k+n,k+n}E_{j+n,i+n} - E_{j+n,i+n}E_{kk} + E_{j+n,i+n}E_{k+n,k+n}) \\
&= \sum \lambda_k (\delta_{ik}E_{ij} - \delta_{jk}E_{ij}) - \lambda_k (\delta_{ik}E_{j+n,i+n} - \delta_{jk}E_{j+n,i+n}) \\
&= (\lambda_i - \lambda_j)(E_{ij} - E_{j+n,i+n})
\end{aligned}$$

Now,

$$\begin{aligned}
[h, E_{i,j+n}] + [h, E_{j,i+n}] &= \sum \lambda_k (E_{kk}E_{i,j+n} - E_{k+n,k+n}E_{i,j+n} - E_{i,j+n}E_{kk} + E_{i,j+n}E_{k+n,k+n}) \\
&+ \sum \lambda_k (E_{kk}E_{j,i+n} - E_{k+n,k+n}E_{j,i+n} - E_{j,i+n}E_{kk} + E_{j,i+n}E_{k+n,k+n}) \\
&= \sum \lambda_k \left(\delta_{ki}E_{i,j+n} + \delta_{kj}E_{i,j+n} + \delta_{kj}E_{j,i+n} + E_{ki}E_{j+n,i+n}\right)
\end{aligned}$$

$$= (\lambda_i + \lambda_j)(E_{i,j+n} + E_{i+n,j})$$

and,

$$[h, E_{i,i+n}] = \sum \lambda_k (E_{kk}E_{i,i+n} - E_{k+n,k+n}E_{i,i+n} - E_{i,i+n}E_{kk} + E_{i,i+n}E_{k+n,k+n})$$
$$= \sum \lambda_k (\delta_{ki}E_{i,i+n} + \delta_{ki}E_{i,i+n})$$
$$= 2\lambda_i E_{i,i+n}$$

□

Problem 7.3. Let $L = o(2n{+}1, \mathbb{F}) = \left\{ x \in gl(2n, \mathbb{F}) : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{pmatrix} x = -x^t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{pmatrix} \right\}$

(1) Write $x = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and find all the conditions on x so that $x \in L$

(2) Let $H = span\{E_{ii} - E_{i+n,i+n} : 2 \le i \le n+1\}$. Show that the root vectors are:
  (a) $\{E_{i+1,j+1} - E_{j+n+1,i+n+1} : 1 \le i \ne j \le n\}$,
  (b) $\{E_{i+1,j+n+1} + E_{j+1,i+n+1} : 1 \le i < j \le n\}$
  (c) $\{E_{1,i+n+1} - E_{i+1,1} : 1 \le i \le n\}$
  (d) $\{E_{1,i+1} - E_{i+n+1,1} : 1 \le i \le n\}$
  (e) $\{E_{i+1,j+1} - E_{j+n+1,i+1} : 1 \le i \ne j \le n\}$

Proof. Let $x = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ where e, f, g, h, i are $n \times n$ matrices, b, c are row vectors, d, g are column vectors and $a \in \mathbb{F}$. Then

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & -I_n & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} a & d^t & g^t \\ b^t & e^t & h^t \\ c^t & f^t & i^t \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -I_n \\ 0 & I_n & 0 \end{pmatrix}$$

gives the following conditions,

(1) $-b^t = g$
(2) $-c^t = d$
(3) $-i^t = e$
(4) $-f^t = f$
(5) $-h^t = h$
(6) $a = -a = 0$ then (5) and (4) give $Tr(x) = 0$. Now the root vectors,
  (a) $\{E_{i+1,j+1} - E_{j+n+1,i+n+1} : 1 \le i \ne j \le n\} \to (\lambda_{i+1} - \lambda_{j+1})$, as in the previous exercise.

(b) $\{E_{i+1,j+n+1} + E_{j+1,i+n+1} : 1 \le i < j \le n\} \to (\lambda_{i+1} + \lambda_{j+1})$, as in the previous exercise.
(c) $\{E_{1,i+n+1} - E_{i+1,1} : 1 \le i \le n\} \to (\lambda_1 - \lambda_{i+1})$
(d) $\{E_{1,i+1} - E_{i+n+1,1} : 1 \le i \le n\} \to (\lambda_{i+1} - \lambda_1)$
(e) $\{E_{i+1,j+1} - E_{j+n+1,i+1} : 1 \le i \ne j \le n\} \to (\lambda_{i+1} + \lambda_{j+1})$.

□

Problem 7.4. Let $L = o(2n, \mathbb{F}) = \left\{ x \in gl(2n, \mathbb{F}) : \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} x = -x^t \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \right\}$

(1) Write $x \in L$ in block form and the termine the conditions on the blocks.
(2) Find an obvious n-dimensional Cartan subalgebra H, the root vectors, and the roots.

Proof. Write $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the conditions for the blocks are $c^t = -b$ and $a^t = -d$.

As in Problem (7.3), we may write $H = span\{E_{ii} - E_{i+n,i+n} : 1 \le i \le n\}$ and the root vectors are:

(1) $\{E_{ij} - E_{j+n,i+n} : 1 \le i \ne j \le n\} \to (\lambda_i - \lambda_j)$.
(2) $\{E_{i,j+n} + E_{j,i+n} : 1 \le i < j \le n\} \to (\lambda_i + \lambda_j)$.
(3) $\{E_{ij} - E_{j+n,i} : 1 \le i \ne j \le n\} \to (\lambda_i - \lambda_j)$.

□

Problem 7.5. Let L be a semisimple Lie algebra and suppose that $x = x_s + x_n$ is the abstract Jordan decomposition of x. Let $\phi : L \to gl(V)$ be a representation of L.

(1) Show that the restriction of $ad(\phi(x_s))$ to $\phi(L)$ is semisimple.
(2) Show that the restriction of $ad(\phi(x_n))$ to $\phi(L)$ is nilpotent.

Proof. (1) Since $ad(x_s)$ is semisimple, L is the sum of eigenspaces of $ad(x_s)$, then $y \in L$ implies that $y = \sum y_i$ where $ad(x_s)y_i = \lambda_i y_i$ with $\lambda_i \in \mathbb{F}$. So $[x_s, y] = \sum \lambda_i y_i$ then $\phi([x_s, y]) = \sum \lambda_i \phi(y_i)$ and so $\phi(L)$ is the sum of eigenspaces of $ad(\phi(x_s))$ hence $ad(\phi(x_s))$ is semisimple.

(2) Since $ad(x_n)$ is nilpotent there exists $m \ge 1$ so that $ad(x_n)^m = 0$ so $ad(\phi(x_n))^m = \phi(ad(x_n)^m) = 0$ then $ad(\phi(x_n))|_{\phi(L)}$ is nilpotent.

□