Elements of Functional Analysis

1. Normed Linear Spaces and Banach Spaces

We will start out with some notation. Let K denote a field either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{X}$ be a linear space over the field K. We will say that $M \subset \mathcal{X}$ is a subspace of $\mathcal{X}$ if it is a subspace of $\mathcal{X}$ as a linear space. Also we will denote by Kx the one dimensional space spanned by the element $x \in \mathcal{X}$ over the field K. If M, N are two subspaces of $\mathcal{X}$ we will denote by $M + N := \{x + y : x \in M, y \in N\}$.

Definition. Let $\mathcal{X}$ be a normed space. Let $\|\cdot\| : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function such that,

- $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in \mathcal{X}$
- $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in \mathcal{X}$ and $\lambda \in K$,

then we call $\|\cdot\|$ is a seminorm on $\mathcal{X}$. If $\|\cdot\|$ is such that $\|x\| = 0$ iff $x = 0$, then we call it a norm, and we call the pair $\mathcal{X}, \|\cdot\|$ a normed linear space.

Proposition 1.1. *If $\mathcal{X}$ is a normed linear space over K, then addition and scalar multiplication are continuous from $\mathcal{X} \times \mathcal{X}$ and $K \times \mathcal{X}$ to $\mathcal{X}$. Moreover, the norm is continuous from $\mathcal{X}$ to $[0, \infty)$; in fact, $|\|x\| - \|y\|| \leq \|x - y\|$.*

Proof. First we will prove that the norm is a continuous function. Then consider $x = (x - y) + y$ and apply the triangle inequality to obtain $\|x\| \leq \|x - y\| + \|y\|$, repeating the same procedure for y we obtain, $|\|x\| - \|y\|| \leq \|x - y\|$. So, given $\epsilon > 0$ if $x, y \in \mathcal{X}$ are such that $\|x - y\| < \epsilon$ then $|\|x\| - \|y\|| < \epsilon$ hence the norm is a continous function.

Now we need to see that the scalar product is continouous then let $\epsilon > 0$ let $x, y \in X$ and $\alpha, \alpha' \in K$ then $\|\alpha x - \alpha' y\| = \|\alpha x - \alpha' x + \alpha' x - \alpha' y\| \leq |\alpha - \alpha'| \|x\| + |\alpha| \|x - y\| < \epsilon$ gives continuity on the scalar and on the vector.

□

Since a norm is just a function defined on a linear space, there can be many norms defined over the same linear space. The question that arrives to mind inmediately is what conditions should two norms have to generate the same topology on the space.

Suppose that two norms $\|\cdot\|_1, \|\cdot\|_2$ generate the same topology on a linear space $\mathcal{X}$. Then consider $B^1_1(0) := \{x \in \mathcal{X} : \|x\|_1 < 1\}$ and $B^2_1(0) := \{x \in \mathcal{X} : \|x\|_2 < 1\}$. Since this two norms define the same topology, then there must be a ball $B^1_r(0)$ around zero, contained in $B^2_1(1)$, and conversely.

Definition. Let $\mathcal{X}$ be a normed linear space and let $\|\cdot\|_1, \|\cdot\|_2$ be two norms defined on $\mathcal{X}$. This norms are said to be equivalent, if there exist two positive constants c_1, c_2 so that $c_1 \|x\|_1 \leq \|x\|_2 \leq c_2 \|x\|_1$ for every $x \in \mathcal{X}$.

From the discussion above, it is clear that two norms generate the same topology on a linear space iff they are equivalent. In this sense, convergent sequences in a linear space with one norm are convergent in the same linear space with an equivalent norm.

Now we will verify that all the norms that can be defined on a finite dimensional linear space are equivalent. So let $\mathcal{X}$ be finite dimensional linear space and let $\{x_1, ..., x_n\}$ be a basis for $\mathcal{X}$. Then every $x \in X$ has a unique representation as $x = \alpha_1 x_1 + ... + \alpha_n x_n$. Define $\|x\| = \sum_{j=1}^{n} |\alpha_j|$, it is easy to verify that this defines a norm on $\mathcal{X}$. Now we will use this norm to compare all the norms that can be defined on $\mathcal{X}$, to this end we will need the following,

Lemma 1.1. *Let $\mathcal{X}$ be a normed linear space with basis $\{x_1, ..., x_n\}$, then there exists $c > 0$ so that $\|x\| = \|\alpha_1 x_1 + ... + \alpha_n x_n\| \geq c \sum_{j=1}^{n} |\alpha_j|$ for every $x \in \mathcal{X}$.*

Proof. Let $s = \sum_{j=1}^{n} |\alpha_j|$, if $s = 0$ the theorem is true for any $c > 0$. Assume $s \neq 0$ then let $\beta_i = \frac{\alpha_i}{s}$, then it suffices to show that $\|\beta_1 x_1 + ... + \beta_n x_n\| \geq c$ whenever $\sum_{j=1}^{n} |\beta_j| = 1$.

For a contradiction, suppose that there exists a sequence $\{y_m\} \subset \mathcal{X}$ where each $y_m = \beta_1^{(m)} x_1 + ... + \beta_n^{(m)} x_n$ so that $\|y_m\| \to 0$ and $\sum_{j=1}^{n} |\beta_j^{(m)}| = 1$, contradicting the existence of $c > 0$ that satisfies the assumption of the lemma.

Notice that $\left\{\beta_j^{(m)}\right\}$ is a bounded set in $\mathbb{C}$ for each $1 \leq j \leq n$, hence it has a convergent subsequence in $\mathbb{C}$. But all this sequences cannot converge to zero, hence $\|y_m\| \to 0$ is not possible. Then, such $c > 0$ must exist. □

Proposition 1.2. *Let $\mathcal{X}$ be finite dimensional linear space with basis $\{x_1, ..., x_n\}$. If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms defined on $\mathcal{X}$ then they are equivalent.*

Proof. Since $x \in \mathcal{X}$ has a unique representation as $x = \alpha_1 x_1 + ... + \alpha_n x_n$ then $\|x\|_1 \leq \sum_{j=1}^{n} |\alpha_j| \|x_n\|_1 \leq k \sum_{j=1}^{n} |\alpha_j|$, where $k = \max \|x_j\|_1$. This result holds for any norm. So it suffices to show that one of the two norms is equivalent to $\|x\| = \sum_{j=1}^{n} |\alpha_j|$. But, this result is inmediate from the previous lemma so this concludes the proof. □

Definition. Let $\mathcal{X}$ be a normed linear space, let $\{x_n\}$ be a sequence of points contained in $\mathcal{X}$. $\{x_n\}$ is said to be a Cauchy sequence if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that for every $n, m > N$ then $\|x_n - x_m\| < \epsilon$. A normed linear space $\mathcal{X}$ is said to be complete or a Banach space if every Cauchy sequence converges in $\mathcal{X}$.

The series $\sum_{n=1}^{\infty} x_n$ converges if there exists an $x \in X$ so that $\sum_{n=1}^{N} x_n \to x$ when $N \to \infty$. It is absolutely convergent iff $\sum_{n=1}^{\infty} \|x_n\| < \infty$.

With our previous lemma, it can also be proved that every finite dimensional normed linear space is a Banach space. The proof of this fact will be presented using the Hahn-Banach theorem in the section on linear functionals. A really handy criterion to evaluate if a normed linear space is complete is stated as follows,

Proposition 1.3. *A normed linear space $\mathcal{X}$ is complete iff every absolutely convergent series converges in $\mathcal{X}$.*

Proof. Suppose $\mathcal{X}$ is a Banach space. Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series. Then let $s_n = \sum_{n=1}^{N} x_n$. We notice that $\{s_n\}$ is a Cauchy sequence, since if we assume WLOG that $n > m$ then $\|s_n - s_m\| = \left\|\sum_{m+1}^{n} x_j\right\| \leq \sum_{m+1}^{n} \|x_j\| \leq \sum_{m+1}^{\infty} \|x_j\| < \epsilon$ for m large enough.

On the other hand, suppose that every absolutely convergent series converges in $\mathcal{X}$, then take a Cauchy sequence $\{x_n\}$ in $\mathcal{X}$. Then for every $j \geq 1 \in \mathbb{Z}$ there exists $N_j \in \mathbb{N}$ so that for every $m, n > N_j$ then $\|x_n - x_m\| < 2^{-j}$. Let $y_1 = x_{N_1}$ and $y_j = x_{N_j} - x_{N_{j-1}}$. Then $\sum \|y_j\| = \sum \left\|x_{N_j} - x_{N_{j-1}}\right\| \leq \|y_1\| + \sum 2^{-j} < \infty$, hence it is absolutely convergent and by hypothesis, convergent. So there exists an $x \in \mathcal{X}$ so that $\sum [x_{N_j} - x_{N_{j-1}}] \to x$. It is easy to see that $x_n \to x$, hence the space is complete. □

Example. A non-trivial application of the criterion exposed above is the following. Let $(\mathcal{X}, \mathcal{M})$ be a measurable space. Let $M(X)$ be the space of all the complex measures defined on $(\mathcal{X}, \mathcal{M})$. For $\mu \in M(X)$ define $\|\mu\| = |\mu|(X)$. It is easy to verify that this is a normed space. To show $M(X)$ is a Banach space, let $\sum_{j=1}^{\infty} \mu_j$ be an absolutely convergent series (i.e. $\sum_{j=1}^{\infty} \|\mu_j\| = \sum_{j=1}^{\infty} |\mu_j|(X) < \infty$). By Lebesgue-Radon-Nikodym theorem $|\mu_j|(X) = \int_X |f_j| d\nu_j$ so that $|f_j| \in L^1(\nu_j)$ then for each $j \geq 1$ we have $\sum \int_X |f_j| d\nu_j = \int_X \sum |f_j| d\nu_j < \infty$, so the series converges in $M(X)$. So, Proposition (1.3) says that $M(X)$ is a Banach space.

1.1. **Linear Operators.**

We will turn our attention to linear operators defined on normed spaces. Let $\mathcal{X}, \|\cdot\|_{\mathcal{X}}$ and $\mathcal{Y}, \|\cdot\|_{\mathcal{Y}}$ be normed linear spaces. The mapping $T : \mathcal{X} \to \mathcal{Y}$ is said to be a linear operator iff for every $x, y \in \mathcal{X}$ and $\lambda \in K$ then $T(x + y) = Tx + Ty$ and $T(\lambda x) = \lambda Tx$. The operator T is said to be bounded if there exists a positive constant C so that $\|Tx\|_{\mathcal{Y}} \leq C \|x\|_{\mathcal{X}}$ for every $x \in \mathcal{X}$. The subindexes on eac norm will be dropped whenever the context is clear.

Definition.(Operator norm) Let $T : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator. The norm of T is defined by $\|T\| = \inf \{C : \|Tx\| \leq C \|x\|\}$. We will denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$.

Proposition 1.4. *Let $T : \mathcal{X} \to \mathcal{Y}$ be a bounded linear operator. If,*

$$\|T\| = \inf \{C : \|Tx\| \leq C \|x\|\}$$

$$\|T\|_1 = \sup \{\|Tx\| : \|x\| = 1\}$$

$$\|T\|_2 = \sup \left\{ \frac{\|Tx\|}{\|x\|} : x \neq 0 \right\}$$

then $\|T\| = \|T\|_1 = \|T\|_2$.

Proof. We will show the following chain of inequalities, $\|T\| \le \|T\|_1 \le \|T\|_2 \le \|T\|$. So by definition of $\|T\|$ we have that $\|Tx\| \le \|T\| \|x\|$. Since this is true for any $x \in \mathcal{X}$ then we can take the supremum over $x \in \mathcal{X}$ so that $\|x\| = 1$, and obtain $\|T\| \le \|T\|_1$.

Now for any non-zero $x \in \mathcal{X}$ consider $\tilde{x} = \frac{x}{\|x\|}$ then $\|T\tilde{x}\| = \frac{\|Tx\|}{\|x\|} \le \|T\|_2$. By taking the supremum over all the $\tilde{x} \in \mathcal{X}$ of norm one on left hand side, we obtain $\|T\|_1 \le \|T\|_2$.

To prove the last part of the chain of inequalities, let $\epsilon > 0$ then by definition of infimum, there exists a non-zero (why?) $x \in X$ so that $\|Tx\| \ge \|T\| \|x\| - \epsilon \|x\|$. So $\frac{\|Tx\|}{\|x\|} \ge \|T\| - \epsilon$. After taking the supremum over the non-zero $x \in \mathcal{X}$ and letting $\epsilon \to 0$, we obtain $\|T\|_2 \le \|T\|$. This gives equality in the chain and completes the proof. ☐

Proposition 1.5. *The space $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ together with the operator norm is a normed linear space.*

Proof. Linearity of the operators give that $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a linear space. Notice that $\|(T+S)x\| = \|Tx + Sx\| \le \|Tx\| + \|Sx\| \le (\|T\| + \|S\|) \|x\|$. Hence,

$$\|T + S\| = \inf_{\|x\|=1} \|(T+S)x\| \le \inf_{\|x\|=1} (\|T\| + \|S\|) \|x\| = \|T\| + \|S\|$$

Consider, $\|(\alpha T)x\| = \|T(\alpha x)\| = \|\alpha \cdot Tx\| = |\alpha| \|Tx\|$. This yields $\|\alpha T\| = |\alpha| \|T\|$. Finally, $\|T\| = 0$ iff $\|Tx\| = 0$ for every $x \in \mathcal{X}$ iff $T \equiv 0$. This completes the proof. ☐

Proposition 1.6. *Given $\mathcal{X}$ and $\mathcal{Y}$ normed linear spaces, let $T : \mathcal{X} \to \mathcal{Y}$ be a linear operator. Then, TFAE:*

(1) $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$
(2) *T is continuous at 0.*
(3) *T is continuous.*

Proof. To see that (1) implies (2), suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ then we have that $\|Tx\| \le \|T\| \|x\|$. So $\|Tx\| < \epsilon$ if $\|x\| < \frac{\epsilon}{\|T\|} = \delta$. This implies that T is continuous at 0. Repeating the argument to $x_1 - x_2 = x$ we obtain that (1) implies (3).

To complete the proof we need to show that (2) implies (1) because (3) implies (2) is trivial. Suppose T is continuous at zero, then there exists $\delta > 0$ such that $\|Tx\| \le 1$ for all non zero $x \in \mathcal{X}$ so that $\|x\| \le \delta$, by linearity, we may apply T to $y = \frac{\delta x}{\|x\|}$, for a non-zero $x \in \mathcal{X}$ then $\|Ty\| = \left\| T\left(\frac{\delta x}{\|x\|}\right) \right\| \le 1$ then $\|Tx\| \le \delta^{-1} \|x\|$ for any $x \in \mathcal{X}$ hence T is bounded. ☐

Proposition 1.7. *Let $\mathcal{X}$ be a normed space and let $\mathcal{Y}$ be a Banach space, then $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a Banach space.*

Proof. Let $\{T_n\}$ be a Cauchy sequence in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ (w.r.t. the operator norm). Then for each $x \in X$ consider $\{T_n x\}$. Since, $\|T_n x - T_m x\| \le \|T_n - T_m\| \|x\|$ and $x \in \mathcal{X}$ is fixed, then we may choose n, m so that $\|T_n - T_m\| < \frac{\epsilon}{\|x\|}$, then for each $x \in \mathcal{X}$, $\{T_n x\}$ is Cauchy in $\mathcal{Y}$, hence convergent in $\mathcal{Y}$. Define $Tx = \lim_{n\to\infty} T_n x$. Consider $\|T_n - T_m\| < \epsilon$ for $m, n > N$ so $\|(T_n - T_m)x\| \leq \epsilon \|x\|$ then by letting $m \to \infty$ we obtain that $\|T_n - T\| < \epsilon \|x\|$, this proves that $\|T_n - T\| \to 0$. Moreover by $|\,\|T_n\| - \|T\|\,| \leq \|T_n - T\| \to 0$ we obtain $\|T\| = \lim \|T_n\|$ hence T is bounded. By linearity of each T_n we have linearity of T. This gives $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and concludes the proof. □

Proposition 1.8. *Let $\mathcal{X}, \mathcal{Y}$ be normed linear spaces. Suppose that $\{T_n\}$ is a sequence in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ and that $T_n \to T$. If $\{x_n\} \subset \mathcal{X}$ and $x_n \to x$, then $T_n x_n \to Tx$.*

Proof. Is left to the reader as an exercise. □

As in the study of functions on the real line, the invertible operators play an important role. We will say that an operator $T : \mathcal{X} \to \mathcal{Y}$ is invertible or an isomorphism, if T is a bijection from $\mathcal{X}$ to $\mathcal{Y}$ and T^{-1} is bounded. If $\|Tx\| = \|x\|$ for every $x \in \mathcal{X}$ is an isometry or an isometric isomorphism.

Proposition 1.9. *Let $\mathcal{X}$ be a Banach space and $T \in \mathcal{L}(\mathcal{X}, \mathcal{X})$. Suppose that $\|I - T\| < 1$ where I is the identity map, then T is invertible. Moreover, if $S \in \mathcal{L}(\mathcal{X}, \mathcal{X})$ is such that $\|S - T\| \leq \left\|T^{-1}\right\|^{-1}$ then S is invertible.*

Proof. Consider an operator $F : \mathcal{X} \to \mathcal{X}$ whose norm is smaller than one. By induction it can be proved that $\|F\|^n = \|F^n\|$, hence

$$\frac{1}{1 - \|F\|} = \sum_{j=0}^{\infty} \|F^n\|$$

Then the series is absolutely convergent in $\mathcal{L}(\mathcal{X}, \mathcal{X})$. Since $\mathcal{X}$ is a Banach space then $\mathcal{L}(\mathcal{X}, \mathcal{X})$ is a Banach space, so by Proposition (1.3) we have that $\sum_{j=0}^{\infty} F^n$ converges in $\mathcal{L}(\mathcal{X}, \mathcal{X})$. Then it is clear that $\|F^n\| \to 0$ when $n \to \infty$, by continuity of the norm then $F^n \to 0$ as $n \to \infty$. Let $x \in \mathcal{X}$ then,

$$\left(\sum_{j=0}^{\infty} F^n\right)(I - F) = \lim_{N\to\infty} \sum_{j=0}^{N} F^n (I - F) = \lim_{N\to\infty} I - F_N = I$$

Then $\sum_{j=0}^{\infty} F^n = (I - F)^{-1}$. Now, it is clear that $(I - F)^{-1}$ is linear, it is bounded, since $I - F$ is bounded and $\|I - F\| \left\|(I - F)^{-1}\right\| \leq 1$, hence $(I - F)^{-1} \in \mathcal{L}(\mathcal{X}, \mathcal{X})$. Now letting $F = I - T$ we obtain the first part of the theorem.

For the second part, notice that the inequality in the assumption implies that $\left\|T^{-1}S - I\right\| \leq \left\|T^{-1}\right\| \|S - T\| \leq 1$, hence $T^{-1}S$ is invertible, this gives S is invertible.

□

Remark. The proposition states that for each invertible operator T, there is a ball contained in $\mathcal{L}(\mathcal{X}, \mathcal{X})$ and centered at T so that every operator in the ball is invertible. Thus, the set of invertible operators is an open set.

1.2. **Construction of Normed Spaces.** The goal in this section is to construct normed linear spaces out of linear spaces with seminorms defined on them. The way to do it will be to force the condition $\|x\| = 0$ iff $x = 0$, by considering a particular quotient space.

First, we consider an easy construction, before getting into matter. Given two normed linear spaces $\mathcal{X}$ and $\mathcal{Y}$ a new normed linear spaced can be constructed, just by considering $\mathcal{X} \times \mathcal{Y} := \{(x, y) : x \in \mathcal{X}, y \in \mathcal{Y}\}$ and defining a norm by,

$$\|(x, y)\| = max(\|x\|, \|y\|)$$

equivalent norms to this can be constructed, e.g. $\|(x, y)\| = \|x\| + \|y\|$ or $\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{\frac{1}{2}}$. This norms are equivalent (Prove!).

Now we turn our attention to the solution of our problem. So first, we will start by considering $M \subset \mathcal{X}$ a closed subspace of the normed linear space $\mathcal{X}$ and define a norm on the quotient space $\mathcal{X}/M = \{x + M : x \in \mathcal{X}\}$ by

$$\|x + M\| = \inf_{y \in M} \|x - y\| \tag{1}$$

Proposition 1.10. *Let $\mathcal{X}$ be a normed linear space and M a closed subspace of $\mathcal{X}$. Then equation (1) defines a norm on $\mathcal{X}/M$.*

Proof. Note that $\|\alpha x + M\| = \inf_{y \in M} \|\alpha x - y\| = \inf_{y \in M} \|\alpha(x - y)\| = |\alpha| \inf_{y \in M} \|x - y\| = |\alpha| \|x + M\|$. Now for the triangle inequality,

$$\begin{aligned}\|z + x + M\| = \inf_{y \in M} \|z + x - y\| &= \inf_{y \in M} \|z + w - y + x - w + y\| \\ &\leq \inf_{y \in M} (\|z + w - y\| + \|x - w + y\|) \\ \leq \inf_{y \in M} \|z + w - y\| + \inf_{y \in M} \|x - w + y\| &= \|z + w + M\| + \|x + w + M\|\end{aligned}$$

Now notice that if $\|z + M\| = 0$ then there is a sequence in M that converges to z, but M is closed, then $z \in M$ hence $z + M = 0$ in the quotient space. The converse is inmediate. So (1) defines a norm on the quotient $\mathcal{X}/M$. $\square$

Lemma 1.2. *Let $\mathcal{X}$ be a normed linear space and M a closed subspace of $\mathcal{X}$ and let $\epsilon > 0$ then there exists $x \in \mathcal{X}$ such that $\|x\| = 1$ and $\|x + M\| \geq 1 - \epsilon$.*

Proof. Let $v \in \mathcal{X} \setminus M$ and let $\rho = \inf_{y \in M} \|v - y\|$. Since M is closed, then we know that $\rho > 0$. By definition of infimum, there exists an $z \in M$ so that $\|v - z\| \leq \rho/(1 - \epsilon)$. Let $c = \|v - z\|^{-1}$ then define $x = c(v - z)$, by construction $\|x\| = 1$. Let $w \in M$ and compute

$$\|x - w\| = \|c(v - z) - w\| = c \left\|v - (z + c^{-1}w)\right\|$$

and note that $z + c^{-1}w \in M$ then $\|x - w\| = c\,\|v - m\|$ for some $m \in M$, then,

$$\|x + M\| = \inf_{y \in M} \|x - y\| = c \inf_{m \in M} \|v - m\| = c\rho = \frac{\rho}{\|v - z\|} \geq \frac{\rho}{\rho/(1-\epsilon)} = 1 - \epsilon$$

□

Now we may consider the projection map $\pi : \mathcal{X} \to \mathcal{X}/M$ defined by $\pi(x) = x + M$. In the light of the previous lemma, $1 - \epsilon \leq \|x + M\| = \|\pi(x)\| \leq \|\pi\|\,\|x\| = \|\pi\|$, letting $\epsilon \to 0$ we get $\|\pi\| \geq 1$. If $x \in M$ then $\|\pi(x)\| = \|x + M\| = \inf\{\|x - y\| : y \in M\} \leq \|x\|$ gives us that $\|\pi\| = 1$. Now, we finally arrive to the last proposition in this section. This will give us a way to construct normed linear spaces out of seminormed linear spaces.

Proposition 1.11. *Let $\mathcal{X}$ be a linear space, $\rho : \mathcal{X} \to [0, \infty)$ a seminorm on $\mathcal{X}$. Let $M = \{x \in X : \rho(x) = 0\}$. Then $\mathcal{X}/M$ with $\|x + M\| = \inf\{\rho(x - y) : y \in M\}$ for every $x + M \in \mathcal{X}/M$ is a normed linear space.*

Proof. Notice that $\|x + M\| = 0$ iff $x \in M$ iff $x + M = 0$ in $\mathcal{X}/M$. □

1.3. **Linear Functionals.** Suppose that $\mathcal{X}$ is a normed linear space over the field K. A special type of operator is a linear operator whose image is contained in K. We will devote the next section to the study of those such operators.

Definition. Let $\mathcal{X}$ is a normed linear space over the field K. The linear map $f : \mathcal{X} \to K$ is called a linear functional. The normed linear space $\mathcal{L}(\mathcal{X}, K)$ will be called the dual space of $\mathcal{X}$ and will be denoted by $\mathcal{X}^*$.

Remark. If $K = \mathbb{C}$ or $\mathbb{R}$ then Proposition (1.7) and the completeness of the field give that $\mathcal{X}^*$ is a Banach space. Then absolutely convergent series of functionals are convergent in $\mathcal{X}^*$.

Proposition 1.12. *Let $\mathcal{X}$ be a normed linear space over $\mathbb{C}$. If $f : \mathcal{X} \to \mathbb{C}$ is a linear functional and $Ref = u$ then u is a real valued linear functional and $f(x) = u(x) - iu(ix)$ for every $x \in X$. Conversely, if u is a real valued linear functional, then $f(x) = u(x) - iu(ix)$ is a complex linear functional. Moreover $\|u\| = \|f\|$*

Proof. Since $Ref(x) = u(x)$ then $Imf(x) = -Re(i \cdot f(x)) = -Ref(ix) = -u(ix)$ then $f(x) = u(x) - iu(ix)$. On the other hand, if u is real valued linear, then f is clearly linear in the real case. Now consider, $f(ix) = u(ix) + iu(x) = i(u(x) - iu(ix)) = if(x)$. Since $\|u\| = \|Ref\| \leq \|f\|$. Now, consider $\alpha = \overline{sgnf(x)}$ then $|f(\alpha x)| = \alpha f(x) = f(\alpha x) = u(\alpha x)$, hence $\|u\| = \|f\|$. □

Definition. Let $\mathcal{X}$ be a real linear space. A sublinear functional $\rho : \mathcal{X} \to \mathbb{R}$ is a function satisfying,

(1) $\rho(x + y) \leq \rho(x) + \rho(y)$ for every $x, y \in \mathcal{X}$,

(2) $\rho(\alpha x) = \alpha\rho(x)$ for $\alpha \geq 0$ and every $x \in \mathcal{X}$.

Theorem 1.3. *(Hahn-Banach)Let $\mathcal{X}$ be a real linear space. Suppose that $\rho : \mathcal{X} \to \mathbb{R}$ is a sublinear functional. Suppose $\mathcal{M}$ is a subspace of $\mathcal{X}$ and f is a linear functional defined on $\mathcal{M}$ satisfying $f(x) \leq p(x)$ for all $x \in \mathcal{M}$. Then, there exists a linear functional $\tilde{f} : \mathcal{X} \to \mathbb{R}$ so that $\tilde{f}|_{\mathcal{M}} = f$ (i.e., $f(x) = \tilde{f}(x)$ for all $x \in \mathcal{M}$) and $\tilde{f}(x) \leq p(x)$ for all $x \in \mathcal{X}$*

Proof. Let $\mathcal{N}$ be the set of all linear extensions g of f so that $g(x) \leq p(x)$ for all $x \in \mathcal{D}(g)$. Clearly $\mathcal{N}$ is not empty, since $f \in \mathcal{N}$. Let $\prec$ be the partial order defined by $g \prec h$ if $\mathcal{D}(g) \subset \mathcal{D}(h)$ and $g(x) = h(x)$ for every $x \in \mathcal{D}(g)$.

Let C be a chain in $\mathcal{N}$, then define $\hat{g} : \mathcal{D}(\hat{g}) \to \mathbb{R}$ where $\mathcal{D}(\hat{g}) = \bigcup_{g \in C} \mathcal{D}(g)$. For every $x \in \mathcal{D}(\hat{g})$ there exists a $g \in C$ so that $x \in \mathcal{D}(g)$, define $\hat{g}(x) = g(x)$. This is well defined, since C is a chain, then if $x \in \mathcal{D}(g')$ for some other $g' \in C$, then either $g \prec g'$ or $g' \prec g$ in either case $g(x) = g'(x) = \hat{g}(x)$. Then, $\hat{g}$ is an upper bound for C and contained in $\mathcal{N}$, so by Zorn's lemma there exists a maximal element, denote it by $\tilde{f}$.

We claim that the domain of $\tilde{f}$ is all of $\mathcal{X}$, since for otherwhise, there exists $y_1 \in \mathcal{X} \backslash \mathcal{D}(\tilde{f})$. Let $Y_1 = span\left\{\mathcal{D}(\tilde{f}), y_1\right\}$, then $x \in Y_1$ is of the form $x = \alpha y_1 + y$ where $\alpha \in \mathbb{R}$ and $y \in \mathcal{D}(\tilde{f})$.

Define $g(\alpha y_1 + y) = \alpha c + \tilde{f}(y)$ with $c \in \mathbb{R}$. It is clear that $\mathcal{D}(\tilde{f}) \subset \mathcal{D}(g)$ and that g is a linear extension of f. If we show that $g(x) \leq p(x)$ for all $x \in \mathcal{D}(g)$ we would contradict the maximality of $\tilde{f}$, and this would give the proof. So, we need to find the appropriate value of c.

Let $y, z \in \mathcal{D}(\tilde{f})$ then

$$\tilde{f}(y) - \tilde{f}(z) = \tilde{f}(y - z) \leq p(y - z) = p(y + y_1 - y_1 - z) \leq p(y + y_1) + p(-y_1 - z)$$

hence

$$-\tilde{f}(z) - p(-y_1 - z) \leq -\tilde{f}(y) + p(y + y_1)$$

Notice that l.h.s does not depend on z and r.h.s does not depend on y and this is true for any $y, z \in \mathcal{D}(\tilde{f})$. Let

$$m_0 = \sup\left\{-\tilde{f}(z) - p(-y_1 - z) : z \in \mathcal{D}(\tilde{f})\right\}$$

$$m_1 = \inf\left\{-\tilde{f}(y) + p(y + y_1) : y \in \mathcal{D}(\tilde{f})\right\}$$

Let $c \in \mathbb{R}$ be so that $m_0 \leq c \leq m_1$ then we have,

$$-\tilde{f}(z) - p(-y_1 - z) \leq c \leq -\tilde{f}(y) + p(y + y_1)$$

With this, we may show that $g(\alpha y + y_1) \leq p(\alpha y + y_1)$. If $\alpha = 0$ there is nothing to prove, since $g(y_1) = \tilde{f}(y_1) \leq p(y_1)$. If $\alpha < 0$ consider,

$$-\tilde{f}\left(\frac{z}{\alpha}\right) - p\left(-\frac{z}{\alpha} - y_1\right) \leq c$$

then, multiplying by α gives,

$$-\alpha p\left(-\frac{z}{\alpha}-y_1\right)\geq \tilde{f}\left(z\right)+\alpha c$$

since $-\alpha>0$ this gives $g(z+\alpha y_1)\leq p(g(z+\alpha y_1))$. If $\alpha>0$ then take the other side of the inequality with y/α,

$$c\leq -\tilde{f}\left(\frac{y}{\alpha}\right)+p\left(\frac{y}{\alpha}+y_1\right)$$

so multiplying by α we obtain,

$$g(y+\alpha y_1)=\tilde{f}(y)+c\alpha\leq p(y+\alpha y_1)$$

Which gives the contradiction we were looking for. This proves that $\mathcal{D}(\tilde{f})=\mathcal{X}$ and completes the proof. $\square$

Theorem 1.4. *(Hahn-Banach for normed linear spaces)Let $Z\subset\mathcal{X}$ be a subspace of the normed linear space $\mathcal{X}$. Let $f:Z\to\mathbb{C}$ be a bounded linear functional. Then there exists a linear extension $\tilde{f}:\mathcal{X}\to\mathbb{C}$ so that $\|f\|_{Z^*}=\left\|\tilde{f}\right\|_{\mathcal{X}^*}$.*

Proof. Since $|f(z)|\leq\|f\|_{Z^*}\|z\|$. Define a sublinear functional by $\rho(x)=\|f\|_{Z^*}\|x\|$ for every $x\in\mathcal{X}$. So, by Hahn-Banach theorem there exists a linear extension $\tilde{f}:\mathcal{X}\to\mathbb{C}$ so that $|\tilde{f}(x)|\leq\rho(x)=\|f\|_{Z^*}\|x\|$ for every $x\in\mathcal{X}$. In particular, for every non-zero $x\in\mathcal{X}$ we have,

$$\frac{|\tilde{f}(x)|}{\|x\|}\leq\|f\|_{Z^*}$$

hence, $\left\|\tilde{f}\right\|_{\mathcal{X}^*}\leq\|f\|_{Z^*}$, the other inequality is clear, then $\|f\|_{Z^*}=\left\|\tilde{f}\right\|_{\mathcal{X}^*}$.

$\square$

Theorem 1.5. *(Hahn-Banach theorem for complex linear spaces) Let $\mathcal{X}$ be a complex linear space. Suppose that $\rho:\mathcal{X}\to[0,\infty)$ is a seminorm. Suppose $\mathcal{M}$ is a subspace of $\mathcal{X}$ and f is a complex linear functional defined on $\mathcal{M}$ satisfying $|f(x)|\leq p(x)$ for all $x\in\mathcal{M}$. Then, there exists a linear functional $\tilde{f}:\mathcal{X}\to\mathbb{C}$ so that $\tilde{f}|_{\mathcal{M}}=f$ (i.e., $f(x)=\tilde{f}(x)$ for all $x\in\mathcal{M}$) and $|\tilde{f}(x)|\leq p(x)$ for all $x\in\mathcal{X}$*

Proof. Let $u=Ref$ by Hahn Banach theorem there exists a linear extension $\tilde{u}:\mathcal{X}\to\mathbb{R}$ so that $\tilde{u}|_{\mathcal{M}}=u$ and $\tilde{u}(x)\leq p(x)$, then define $\tilde{f}(x)=\tilde{u}(x)-i\tilde{u}(ix)$ as in Proposition (1.12). Let $\alpha=sgn\tilde{f}(x)$ then $|\tilde{f}(\alpha x)|=\alpha\tilde{f}(x)=\tilde{f}(\alpha x)=\tilde{u}(\alpha x)\leq p(\alpha x)=|\alpha|p(x)=p(x)$. $\square$

Proposition 1.13.
1. *Let $\mathcal{X}$ be a normed linear space, $\mathcal{M}$ a subspace of $\mathcal{X}$. Then there exists $x_0\in\mathcal{X}$ so that $\delta=\inf_{y\in\mathcal{M}}\|x_0-y\|>0$ and $f\in\mathcal{X}^*$ with $\|f\|=1$ such that $f(x_0)=\delta$ and $f(y)=0$ for every $y\in\mathcal{M}$.*
2. *If $x_0\neq0$ then there exists $f\in\mathcal{X}^*$ such that $\|f\|=1$ and $f(x_0)=\|x_0\|$.*
3. *The bounded linear functionals on $\mathcal{X}$ separate points.*

Proof. (1) Consider $Y = span\{\mathcal{M}, x_0\}$ then $y \in Y$ is of the form $y = \lambda x_0 + z$ where $\lambda \in K$ and $z \in \mathcal{M}$. So, define $f(\lambda x_0 + z) = \lambda\delta$ then f is linear and $f|_{\mathcal{M}} = 0$ and $f(x_0) = \delta$. To see that $\|f\| = 1$, suppose $y \notin \mathcal{M}$ then $\lambda \neq 0$ then $|f(y)| = |f(\lambda x_0 + z)| = |\lambda|\delta \leq |\lambda| \left\|x_0 + \frac{z}{\lambda}\right\| = \|\lambda x_0 + z\| = \|y\|$ so $\|f\| \leq 1$. Now, using $p(x) = \|x\|$ we can apply Hahn-Banach theorem to obtain the desired result.

(2) Follows from (1) by considering $\mathcal{M} = \{0\}$.

(3) If $x \neq y$ then $x - y \neq 0$ then there exists a functional f so that $f(x-y) \neq 0$, then $f(x) \neq f(y)$.

□

Given a normed linear space $\mathcal{X}$, for each $x \in \mathcal{X}$ define a functional $\hat{x} : \mathcal{X}^* \to \mathbb{C}$ as $\hat{x}(f) = f(x)$ for every $f \in \mathcal{X}^*$. It is clear that $\hat{x}$ is linear and $|\hat{x}(f)| = |f(x)| \leq \|f\|_{\mathcal{X}^*} \|x\|$ hence $\|\hat{x}\| \leq \|x\|$, and part (2) of the previous theorem implies $\|\hat{x}\| = \|x\|$.

Proposition 1.14. *Let $\mathcal{X}$ be a complex normed linear space and $\mathcal{M}$ be a closed subspace of $\mathcal{X}$. If $x \in \mathcal{X} \setminus \mathcal{M}$ then $\mathcal{M} + \mathbb{C}x$ is closed. In particular, every finite-dimensional subspace of $\mathcal{X}$ is closed, moreover, every finite-dimensional normed linear space is a Banach space.*

Proof. Let $\{m_j + \alpha_j x\}_{j \geq 1}$ be a Cauchy sequence in $\mathcal{M} + \mathbb{C}x$. By Proposition(1.13) there exists a functional $f \in \mathcal{X}^*$ with $\|f\| = 1$, so that $f|_{\mathcal{M}} = 0$ and $f(m + \alpha x) = \alpha$ for every $m + \alpha x \in \mathcal{M} + \mathbb{C}x$ then $|f(m_n + \alpha_n x) - f(m_k + \alpha_k x)| = |\alpha_n - \alpha_k| \leq \|f\| \, \|m_n + \alpha_n x - (m_k + \alpha_k x)\| < \epsilon$ for n, k large enought. Then $\{\alpha_j\}$ is a Cauchy sequence in $\mathbb{C}$ hence convergent, thus $\{m_j\}$ is Cauchy in $\mathcal{M}$ which is closed, hence convergent. Let m and α be the limits of $\{m_j\}$ and $\{\alpha_j\}$ respectively. Then, it is easy to see that $\{m_j + \alpha_j x\}$ converges to $m + \alpha x$. Now, proceeding by induction, we obtain the rest of the proposition. □

1.4. Baire Category Theorem.

In this section, we will present the Baire Category Theorem and some of its applications in functional analysis. In particular, will allow us to prove theorems about operators defined on Banach spaces.

Definition. Let $(\mathcal{X}, \tau)$ be a topological space. A subset $A \subset \mathcal{X}$ is nowhere dense if $\overline{A}$ does not contain any open set (i.e. $(\overline{A})^\circ = \emptyset$, other way to see it is $\mathcal{X} \setminus A$ is dense in $\mathcal{X}$)

Theorem 1.6. *(The Baire Category Theorem) Let X be a complete metric space.*

1. *If $\{U_n\}_{n \geq 1}$ is a sequence of open dense subsets of X, then $\bigcap_1^\infty U_n$ is dense in X.*
2. *X is not a countable union of nowhere dense sets.*

Remark. Another way to write (2) is that if $X = \bigcup_1^\infty U_n$ then there exists $k \geq 1$ so that $(\overline{U_k})^\circ \neq \emptyset$. This is the way in which we will use it most of the times.

Proof. We may show that given an open set $W \subset X$ then $W \cap (\bigcap_1^\infty U_n) \neq \emptyset$. To see this, notice that $W \cap U_1 \neq \emptyset$ since U_1 is dense in X, since both sets are open, its intersection is open, hence there exists a ball $B(x_0, r_0)$, where we may assume that $0 < r_0 < 1$. Inductively, we may choose an open ball $B(x_n, r_n)$ for $n > 1$ by noticing that $U_n \cap B(x_{n-1}, r_{n-1}) \neq \emptyset$ since U_n is dense, then we can choose $B(x_n, r_n)$ so that $0 < r_n < 2^{-n}$ and $\overline{B(x_n, r_n)} \subset U_n \cap B(x_{n-1}, r_{n-1})$. In this fashion, we have constructed a Cauchy sequence $\{x_n\}$, since X is complete, the sequence is convergent, then $x \in \overline{B(x_N, r_N)} \subset U_N \cap B(x_{N-1}, r_{N-1}) \subset U_N \cap W$ for all N, this completes the proof for (1).

Part (2) of the theorem follows from considering a sequence of nowhere dense sets $\{X_n\}$ then $\left\{(\overline{X_n})^c\right\}$ is a sequence of open dense sets, so by (1) its intersection is dense, in particular $\bigcap_1^\infty (\overline{X_n})^c \neq \emptyset$, then $\bigcup_1^\infty X_n \subset \bigcup_1^\infty \overline{X_n} = \left(\bigcap_1^\infty (\overline{X_n})^c\right)^c \neq X$. ☐

Before begining with the following lemma, we need some notation let B_r refer to a ball of radius r around zero.

Lemma 1.7. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be surjective, then there exists $r > 0$ so that $B_r \subset T(B_1)$.*

Proof. Notice $X = \bigcup_1^\infty \overline{B_{k/2}}$, then T surjective implies that $Y = \bigcup_1^\infty \overline{T(B_{k/2})} = \bigcup_1^\infty k\overline{T(B_{1/2})}$. Since $\mathcal{Y}$ is complete, the Baire Category Theorem says that there exists $k \geq 1$ so that $\left(\overline{T(B_{k/2})}\right)^\circ \neq \emptyset$.

Then there exists a $y_0 \in \overline{T(B_{k/2})}$ and $\delta > 0$ so that $B_\delta(y_0) \subset \overline{T(B_{k/2})}$, if we let $\epsilon = \delta/k$ then $B_\epsilon(y_0) \subset \overline{T(B_{1/2})}$, so $B_\epsilon \subset \overline{T(B_{1/2})} - y_0 \subset \overline{T(B_1)}$.

So far, we have constructed a ball of radius ϵ that is contained in $\overline{T(B_1)}$, we still need to shrink this ball to fit not only in the closure, but in $T(B_1)$ itself.

To this end, consider $y \in B_{\epsilon/2}$ then there exists $x_1 \in B_{1/2}$ so that $\|y - Tx_1\| < \epsilon/4$, if we continue this process, we may find $x_n \in B_{1/2^n}$ such that $\|y - \sum_1^n Tx_j\| < \epsilon/2^n$ for each $n \in \mathbb{N}$. In this fashion, we can construct a Cauchy sequence in $\mathcal{X}$, since $\mathcal{X}$ is a Banach space, then there exists $x \in \mathcal{X}$ so that $\sum x_n \to x$. Now, notice that $\|x\| < \sum 2^{-n} = 1$ and that $y = Tx$. But, $y \in B_{\epsilon/2}$ was arbitrary, so $B_{\epsilon/2} \subset T(B_1)$. Letting $r = \epsilon/2$ we obtain the proof of the lemma. ☐

Definition. Let X and Y be topological spaces, the mapping $f : X \to Y$ is said to be open if $f(U)$ is open in Y whenever U is open in X.

Notice that in the case, in which X and Y are metric spaces. This means that $f : X \to Y$ is open iff for every open ball B and every $x \in B$ there exists a ball centered at $f(x)$ and contained in $f(B)$. If $f : X \to Y$ is linear, then f conmutes with translations and dilations, the previous lemma gives the implication of the following,

Theorem 1.8. *(Open mapping theorem) Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is surjective iff T is open.*

Proof. For the converse, suppose T is open then ☐

Corollary 1.9. *If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a bijection, then T is a isomorphism.*

Proof. We need T^{-1} continuous. But, by Theorem (1.8), T is open then T^{-1} is continuous. □

Definition. Given $\mathcal{X}$, $\mathcal{Y}$ normed linear spaces and $T : \mathcal{X} \to \mathcal{Y}$ a linear map. Then the graph of T is defined by $\Gamma(T) := \{(x, Tx) \in \mathcal{X} \times \mathcal{Y} : x \in \mathcal{X}\}$. Moreover, T is said to be a closed map if $\Gamma(T)$ is a closed subspace of $\mathcal{X} \times \mathcal{Y}$.

Theorem 1.10. *(Closed Graph Theorem) Given $\mathcal{X}$, $\mathcal{Y}$ normed linear spaces and the linear map $T : \mathcal{X} \to \mathcal{Y}$ is closed then T is bounded.*

Proof. Since, $\mathcal{X}$ and $\mathcal{Y}$ are complete, so is $\mathcal{X} \times \mathcal{Y}$, then $\Gamma(T)$ is a closed subspace of a Banach space, hence complete. Let $\pi_{\mathcal{X}} : \Gamma(T) \to \mathcal{X}$ and $\pi_{\mathcal{Y}} : \Gamma(T) \to \mathcal{Y}$ be the projection mappings onto $\mathcal{X}$ and $\mathcal{Y}$. We have proved before that the projection mappings are linear and bounded of norm one. So, by Corollary (1.9), $\pi_{\mathcal{X}}$ is an isomorphism from $\Gamma(T)$ to $\mathcal{X}$, hence invertible and the inverse is bounded. Now observing $T = \pi_{\mathcal{Y}} \circ \pi_{\mathcal{X}}^{-1}$ gives that T is bounded. □

Definition. Let $\mathcal{X}$ be a topological space. $E \subset \mathcal{X}$ is said to be meager if it is the countable union of nowhere dense sets.
Remark. From Baire category theorem we have that every Banach space $\mathcal{X}$ is nonmeager.

Theorem 1.11. *(The Uniform Boundedness Principle) Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed linear spaces and $\mathcal{A}$ is a subset of $\mathcal{L}(\mathcal{X}, \mathcal{Y})$. If $\sup_{T \in \mathcal{A}} \|Tx\| < \infty$ for all x in some nonmeager subset $E \subset \mathcal{X}$ (possibly all $\mathcal{X}$) then $\sup_{T \in \mathcal{A}} \|T\| < \infty$.*

Proof. Let $E_n = \{x \in \mathcal{X} : \sup_{T \in \mathcal{A}} \|Tx\| \leq n\} = \bigcap_{T \in \mathcal{A}} \{x \in \mathcal{X} : \|Tx\| \leq n\}$ be closed sets in $\mathcal{X}$. By assumption of the theorem there exists $k \geq 1$ for which $E_k^\circ = (\overline{E_k})^\circ \neq \emptyset$. Then, there exists a point $y \in \mathcal{X}$ and $r > 0$ so that $B_r(y) \subset E_k$. Notice that if $\|x - y\| < r$ then $\|Tx\| \leq \|T(x - y)\| + \|Ty\| \leq 2k$, hence $B_r \subset E_{2k}$. This is true whenever $\|x\| < r$ hence $\sup_{T \in \mathcal{A}} \|T\| < \frac{2k}{r}$. This completes the proof. □

1.5. **Applications.** In this section, we will expose some examples of the applications of the theorems in the previous section. First, we will see a nonexample of the closed graph theorem.

That is an example when $\mathcal{X}$ is not complete, then a mapping is closed but not bounded. The operator defined will be a really well know operator $T = \frac{d}{dx}$ which is not continuous under the uniform norm, of this problem we will take care in the next section on Fréchet spaces.

Suppose $\mathcal{X} = C^1([0, 1])$ and $\mathcal{Y} = C([0, 1])$ with the norm $\|f\| = sup|f(x)|$ in each space. Let $T : \mathcal{X} \to \mathcal{Y}$ be defined by $Tf = -if'$. Notice $\mathcal{X}$ is not complete with this norm, since that polynoms are dense in $\mathcal{X}$. Then let $\{f_n\}$ be so that $f_n \to f$ in $\mathcal{X}$ where $f(x) = |x - \frac{1}{2}|$. Clearly $f_n \to f$ and $f \notin \mathcal{X}$, hence $\mathcal{X}$ is not complete.

So, the conclusion of the closed graph theorem needs not to hold. Consider $\Gamma(T) := \{(f, Tf) \in \mathcal{X} \times \mathcal{Y} : f \in \mathcal{X}\}$ with the norm defined as $\|f\|_{\Gamma(T)} = \|f\| + \|f'\|$. To see that $\Gamma(T)$ is closed, we may show that this norm makes $\mathcal{X}$. In general, something stronger is true.

Proposition 1.15. *Let $\mathcal{X} = C^k([0,1])$ and the norm $\|\cdot\| : \mathcal{X} \to [0,\infty)$ be defined by $\|f\| = \sum_1^k \left\|f^{(k)}\right\|$. Then $\mathcal{X}, \|\cdot\|$ is a Banach space.*

Proof. Let $\mathcal{X}, \|\cdot\|$ be defined as in the assumption of the proposition and let $\sum_{j\geq 1} \|f_j\|$ be an absolutely convergent series on $\mathcal{X}$. Notice that,

$$\sum_{j\geq 1} \|f_j\| = \sum_{j\geq 1}\sum_{i=0}^{k} \left\|f_j^{(i)}\right\| = \sum_{i=0}^{k}\sum_{j\geq 1} \left\|f_j^{(i)}\right\| < \infty$$

Then $\sum_{j\geq 1} \left\|f_j^{(i)}\right\| < \infty$ for each $0 \leq i \leq k$, so it suffices to show for $k = 1$, then induction on k gives the general result.

Now, suppose that $\{f_j\}_{j\geq 1}$ is a Cauchy sequence on $C^1([0,1])$, then for $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that $\|f_n - f_m\| = \sup_{x\in\mathcal{X}} |f_n(x) - f_m(x)| + \sup_{x\in\mathcal{X}} |f_n'(x) - f_m'(x)| < \epsilon$ whenever $m, n > N$, letting $m \to \infty$, this means that $f_n' \to g$ uniformly, and $f_n \to f$ uniformly and $f \in C^1([0,1])$. Now, we need to prove that $f' = g$.

Then, consider that $\int_0^x f_n'(t)dt = f_n(x) - f_n(0)$ if we take limits in both sides, then by Dominated convergence theorem we get $\int_0^x g(t)dt = \int_0^x \lim f_n'(t)dt = f(x) - f(0)$ hence $f' = g$ and the proposition is proves. □

Proposition 1.16. *If $\mathcal{X}$ is a complete normed linear space with $\|\cdot\|_1$ and with $\|\cdot\|_2$. If $\|\cdot\|_1 \leq \|\cdot\|_2$ then the norms are equivalent.*

Proof. We need to find a constant $c > 0$ so that $\|x\|_2 \leq c\|x\|_1$ for all $x \in \mathcal{X}$. So, denote by $\mathcal{X}_1$ the space $\mathcal{X}, \|\cdot\|_1$ and by $\mathcal{X}_2$ the space $\mathcal{X}, \|\cdot\|_2$ and let $T : \mathcal{X}_1 \to \mathcal{X}_2$ be defined by $Tx = x$ since $\Gamma(T)$ is closed by assumption that $\mathcal{X}_1$ and $\mathcal{X}_2$ are complete spaces and T is the identity map. Then Theorem (1.11) says that T is bounded. Then $\|x\|_2 = \|Tx\|_2 \leq \|T\|\|x\|_1$ for every $x \in \mathcal{X}$ and $\|T\| < \infty$. Then, the norms are equivalent. □

Proposition 1.17. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $\{T_n\}$ be a sequence in $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that $\lim T_n x$ exists for every $x \in \mathcal{X}$. Let $Tx = \lim T_n x$ then $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$*

Proof. Linearity of T follows by continuity and linearity of each T_n. We may show only boundness. Since $\{\|T_n x\|\}_{n\geq 1}$ is a convergent sequence on $[0,\infty)$ for each $x \in \mathcal{X}$ then is bounded, hence $sup_n \|T_n x\| < \infty$, then Theorem (1.11) gives that $sup_n \|T_n\| < \infty$ then $\|T\| < \infty$. □

Proposition 1.18. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, $T : \mathcal{X} \to \mathcal{Y}$ a linear operator such that $f \circ T \in \mathcal{X}^*$ for every $f \in \mathcal{X}^*$ then T is bounded.*

Proof. Let $\mathcal{A} = \{f \in \mathcal{X}^* : \|f\| \leq 1\}$ and let $x \in \mathcal{X}$ be fixed. Then consider $\|f \circ Tx\| \leq \|f \circ T\| \|x\| < \infty$ for each $f \in \mathcal{A}$. By Theorem (1.11) we have that $\sup_{f\in\mathcal{A}} \|f \circ T\| = \|T\| < \infty$. ☐

2. Hilbert Spaces

In this section we will study a more specific type of space. This are inner product spaces or pre-Hilbert spaces.

Definition. Let $\mathcal{H}$ be a linear space over the field K, let $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \to K$ be a map so that, for every $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in K$,

- $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$,
- $\langle x, y \rangle = \overline{\langle y, x \rangle}$ and,
- $\langle x, x \rangle \in (0, \infty)$ for $x \neq 0$.

then, $\langle \cdot, \cdot \rangle$ is called and inner product and $\mathcal{H}, \langle \cdot, \cdot \rangle$ is said to be an inner product space or a pre-Hilbert space.

Proposition 2.1. *(Schwarz Inequality) Let $\mathcal{H}$ be a pre-Hilbert space, then $| \langle x, y \rangle | \leq \|x| \|y|$ for all $x, y \in \mathcal{H}$. Equality holds iff x and y are linearly dependent.*

Proof. If $| \langle x, y \rangle | = 0$ there is nothing to prove. So, suppose that $| \langle x, y \rangle | \neq 0$, then by definition, $x \neq 0$ and $y \neq 0$. So, let $\alpha = sgn \langle x, y \rangle$ and $z = \alpha y$, then $\langle x, z \rangle = \langle z, x \rangle = \langle x, y \rangle$ so for $t \in \mathbb{R}$,

$$0 \leq \langle x - tz, x - tz \rangle = \|x\|^2 - 2t \langle x, z \rangle + t^2 \|z\|^2 \tag{2}$$

Taking the derivative with respect to t and setting equal to zero we obtain the maximum which is, $t = \frac{\langle x,z \rangle}{\|z\|^2}$. Substituting in (2) and multiplying everything by $\|z\|^2$ we obtain:

$$0 \leq \|x\|^2 \|z\|^2 - 2 \langle x, z \rangle^2 + \langle x, z \rangle^2 = \|x\|^2 \|z\|^2 - \langle x, z \rangle^2$$

Which gives the result, since $\|y\| = \|z\|$. Equality holds iff $x - tz = 0$ iff $x = \alpha ty$. ☐

Proposition 2.2. *Let $\mathcal{H}$ be a pre-Hilbert space. Then $x \mapsto \sqrt{\langle x, x \rangle} = \|x\|$ is a norm on $\mathcal{H}$.*

Proof. Notice that $\sqrt{\langle x, x \rangle} = 0$ iff $x = 0$ by linearity, and $\|\lambda x\| = \sqrt{\langle \lambda x, \lambda x \rangle} = \sqrt{\lambda \overline{\lambda} \langle x, x \rangle} = |\lambda| \sqrt{\langle x, x \rangle} = |\lambda| \|x\|$. So, the only thing to prove is the triangle inequality. Consider

$$\begin{aligned} \|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + 2Re \langle x, y \rangle + \|y\|^2 \leq \|x\|^2 + 2 \|x\| \|y\| + \|y\|^2 \\ = (\|x\| + \|y\|)^2 \end{aligned}$$

Hence $\|\cdot\|$ is a norm. ☐

So every inner product space is a normed space. But not every norm comes from a inner product, as we shall prove later. Now that we have a notion of a norm in the space, we may talk about Cauchy sequences in the space.

Definition. We say that the inner product space $\mathcal{H}$ is a Hilbert space if every Cauchy sequence is convergent in $\mathcal{H}$.

Example. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $L^2(\mu)$ be the set of equivalent classes of measurable, a.e. equal functions, such that $\int_X |f|^2 d\mu < \infty$. Since $ab \leq \frac{1}{2}(a^2 + b^2)$ then $f, g \in L^2(\mu)$ implies that $|f\overline{g}| \in L^1(\mu)$. So, define $\langle f, g \rangle = \int_X |f\overline{g}| d\mu$. It is easy to verify that this defines an inner product on $L^2(\mu)$.

Let $\{f_n\}$ be a Cauchy sequence on $L^2(\mu)$. (i.e. for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ so that $\|f_n - f_m\| = \left[\int_X |f_n - f_m|^2 d\mu\right]^{1/2} < \epsilon$ whenever $m, n > N$). In the following section, we will prove that this space is complete. If μ is the counting measure, then we denote the space by $\ell_2(A) := \left\{f : A \to \mathbb{C} : \sum_{\alpha \in A} |f(\alpha)|^2 < \infty\right\}$, and the inner product becomes $\langle f, g \rangle = \int_X |f\overline{g}| d\mu = \sum_{\alpha \in A} f(\alpha)\overline{g(\alpha)}$. Notice that this sum can contain at most countable non-zero terms. As well as $L^2(\mu)$, $\ell_2(A)$ is a Hilbert space.

Proposition 2.3. *Let $\mathcal{H}$ be an inner product space, $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ sequences on $\mathcal{H}$ so that $x_n \to x$ and $y_n \to y$ then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.*

Proof. Notice that $|\langle x_n, y_n \rangle - \langle x, y \rangle| = |\langle x_n, y_n \rangle - \langle x_n, y \rangle + \langle x_n, y \rangle - \langle x, y \rangle| \leq |\langle x_n, y_n - y \rangle| + |\langle x_n - x, y \rangle|$ Now by Schwarz inequality,

$$|\langle x_n, y_n \rangle - \langle x, y \rangle| \leq \|x_n\| \, \|y_n - y\| + \|y\| \, \|x_n - x\| \to 0$$

By continuity of the norm. □

Proposition 2.4. *(The Parallelogram Law) For all $x, y \in \mathcal{H}$ with the norm induced by the inner product,*

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

Proof. Just add, $\|x \pm y\|^2 = \|x\|^2 \pm 2Re \langle x, y \rangle + \|y\|^2$ □

Definition. Let $\mathcal{H}$ be an inner product space. $x, y \in \mathcal{H}$ are said to be orthogonal if $\langle x, y \rangle = 0$ and we write it as $x \perp y$. For $E \subset \mathcal{H}$ a subspace, we call the set $E^\perp := \{x \in \mathcal{H} : \langle x, y \rangle = 0$ for every $y \in E\}$ the orthogonal subspace of E.

If $\mathcal{H}$ is a Hilbert space Proposition (2.3) implies that $E^\perp$ is a closed subspace of $\mathcal{H}$.

Theorem 2.1. *If $x_1, ..., x_n \in \mathcal{H}$ and $x_i \perp x_j$ for all $i \neq j$ then,*

$$\left\|\sum_{j=1}^n x_j\right\|^2 = \sum_{j=1}^n \|x_j\|^2$$

Proof. Since $\langle x_i, x_j \rangle = 0$ for all $i \neq j$ then,

$$\left\| \sum_{j=1}^{n} x_j \right\|^2 = \left\langle \sum_{j=1}^{n} x_j, \sum_{j=1}^{n} x_j \right\rangle = \sum_{j=1}^{n} \langle x_j, x_j \rangle = \sum_{j=1}^{n} \|x_j\|^2$$

□

Theorem 2.2. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. Then every $x \in \mathcal{H}$ can be written uniquely as $x = y + z$ where $y \in \mathcal{M}$ and $z \in \mathcal{M}^{\perp}$, where the distances from y and z to x are minimal.*

Proof. Consider $\delta = \inf_{y \in \mathcal{M}} \|x - y\|$ and let $\{y_n\}$ be a sequence in $\mathcal{M}$ so that $\|x - y_n\| \to \delta$. So, by applying the parallelogram law to $y_n - x$ and $y_m - x$ we have,

$$\|y_n + y_m - 2x\|^2 + \|y_n - y_m\|^2 = 2\left(\|y_n - x\|^2 + \|y_m - x\|^2\right)$$

then,

$$\begin{aligned} \|y_n - y_m\|^2 &= 2\left(\|y_n - x\|^2 + \|y_m - x\|^2\right) - 4\left\|\frac{1}{2}(y_n + y_m) - x\right\|^2 \\ &\leq 2\left(\|y_n - x\|^2 + \|y_m - x\|^2\right) - 4\delta^2 \to 0 \end{aligned}$$

This proves that $\{y_n\}$ is a Cauchy sequence in $\mathcal{M}$ which is closed, then there exists $y \in \mathcal{M}$ so that $y = \lim y_n$. Let $z = x - y$ then to complete the proof we need to show that $z \in\in \mathcal{M}^{\perp}$.

So, let $v \in \mathcal{M}$ be so that $\langle v, z \rangle \in \mathbb{R}$ then the function $f(t) = \|z + tu\|^2 = \|z\|^2 + 2t\langle z, u \rangle + t^2\|u\|^2$ then notice that $z - tu = x - (y + tu)$ where $y + tu \in \mathcal{M}$ then $f(0) = \|x - y\|^2 = \delta^2$ is a minimum for $f(t)$, hence $f'(0) = 0 = 2\langle z, u \rangle$ this is true for every $u \in \mathcal{M}$ hence $z \in \mathcal{M}^{\perp}$. Uniqueness follows from considering $x = y + z = y' + z'$ then $y - y' = z' - z$ where $y - y' \in \mathcal{M}$ and $z' - z \in \mathcal{M}^{\perp}$ but $\mathcal{M} \cap \mathcal{M}^{\perp} = \{0\}$, so $z = z'$ and $y = y'$. □

Theorem 2.3. *(Riez's Representation Theorem for linear functionals) If $f \in \mathcal{H}^*$, there is a unique $y \in \mathcal{H}$ so that $f(x) = \langle x, y \rangle$ for all $x \in \mathcal{X}$.*

Proof. If $f = 0$ then $y = 0$ satisfy the theorem, so suppose $f \neq 0$. Since, $f \neq 0$ then $\mathcal{M} = Kerf \neq \mathcal{H}$, $\mathcal{M}$ is a closed subspace then $\mathcal{M}^{\perp} \neq \{0\}$, by Theorem (2.2). Let $z \in \mathcal{M}^{\perp}$ so that $\|z\| = 1$. Notice that $u = f(x)z - f(z)x \in \mathcal{M}$, because $f(u) = 0$, then $0 = \langle u, z \rangle = \langle f(x)z - f(z)x, z \rangle = f(x)\|z\|^2 - \left\langle x, \overline{f(z)}z \right\rangle = f(x) - \left\langle x, \overline{f(z)}z \right\rangle$. Then $y = \overline{f(z)}z$ gives the result. To see the uniqueness, suppose that $\langle x, y \rangle = \langle x, y' \rangle$ for all $x \in \mathcal{H}$ then $\langle x, y - y' \rangle = 0$. By letting $x = y - y'$ we obtain $\|y - y'\| = 0$, so $y = y'$. □

Definition. Let $U = \{u_\alpha\}_{\alpha \in \mathcal{A}}$ be a subset of $\mathcal{H}$ so that $\langle u_\alpha, u_\beta \rangle = 0$ if $\alpha \neq \beta$, then U is a orthogonal set. Moreover, if $\|u_\alpha\| = 1$ for each $\alpha \in \mathcal{A}$, then U is said to be an orthonormal set. If U is an orthonormal set and $\overline{spanU} = \mathcal{H}$ then U is called a total orthonormal set.

Proposition 2.5. *(Bessel's Inequalty) If* $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ *is an orthonormal set in* $\mathcal{H}$*, then for* $x \in \mathcal{H}$ *we have,*

$$\sum_{\alpha\in\mathcal{A}} |\left\langle x, u_\alpha \right\rangle|^2 \le \|x\|^2$$

Proof.

$$0 \le \left\| x - \sum_{\alpha\in\mathcal{A}} \left\langle x, u_\alpha \right\rangle u_\alpha \right\|^2 = \|x\|^2 - 2Re \left\langle x, \sum_{\alpha\in\mathcal{A}} \left\langle x, u_\alpha \right\rangle u_\alpha \right\rangle + \left\| \sum_{\alpha\in\mathcal{A}} \left\langle x, u_\alpha \right\rangle u_\alpha \right\|^2$$
$$\le \|x\|^2 - 2\sum_{\alpha\in\mathcal{A}} |\left\langle x, u_\alpha \right\rangle|^2 + \sum_{\alpha\in\mathcal{A}} \|\left\langle x, u_\alpha \right\rangle u_\alpha\|^2 = \|x\|^2 - \sum_{\alpha\in\mathcal{A}} |\left\langle x, u_\alpha \right\rangle|^2$$

Which gives the proof. ☐

Notice, that in Bessel's inequality at most countably many $\langle x, u_\alpha\rangle$'s can be non-zero.

Theorem 2.4. *Let* $\{u_\alpha\}_{\alpha\in\mathcal{A}}$ *be a orthonormal set in a Hilbert space* $\mathcal{H}$*. Then* $x = \sum_{\alpha\in\mathcal{A}} \left\langle x, u_\alpha \right\rangle u_\alpha$ *for each* $x \in \mathcal{H}$ *iff for each non-zero* $x \in \mathcal{H}$ *there exists* $\alpha \in \mathcal{A}$ *so that* $\left\langle x, u_\alpha \right\rangle \ne 0$ *iff* $\|x\|^2 = \sum_{\alpha\in\mathcal{A}} |\left\langle x, u_\alpha \right\rangle|^2$ *for each* $x \in \mathcal{H}$*.*

Proof. The first implication is trivial. Now consider

$$\left\| \sum_{j=m+1}^{n} \left\langle x, u_{\alpha_j} \right\rangle u_{\alpha_j} \right\|^2 = \sum_{j=m+1}^{n} |\left\langle x, u_{\alpha_j} \right\rangle|^2 \to 0$$

by Bessel's inequality. Then the series converges absolutely, but $\mathcal{H}$ is complete, so the series $\sum_{\alpha\in\mathcal{A}} \left\langle x, u_\alpha \right\rangle u_\alpha$ converges. Now consider $y = x - \sum_{\alpha\in\mathcal{A}} \left\langle x, u_\alpha \right\rangle u_\alpha$ and compute $\left\langle y, u_\alpha \right\rangle = \left\langle x, u_\alpha \right\rangle - \left\langle \sum_{\beta\in\mathcal{A}} \left\langle x, u_\beta \right\rangle u_\beta, u_\alpha \right\rangle = \left\langle x, u_\alpha \right\rangle - \sum_{\beta\in\mathcal{A}} \left\langle x, u_\beta \right\rangle \left\langle u_\beta, u_\alpha \right\rangle = 0$ but by assumption we have that $y = 0$ this gives that $x = \sum_{\alpha\in\mathcal{A}} \left\langle x, u_\alpha \right\rangle u_\alpha$ ☐