# MEASURES

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ABSTRACT. The following is a summary of the material from the first chapter of Folland G. "Real Analysis Modern Technics and Their Applications".

## 1. $\sigma$-ALGEBRAS

This section covers the sets that will serve as the domain of our functions called measures.

**Definition 1.** *(Algebra) Let X be any set, an algebra $\mathcal{A}$ is a collection of subsets of X, so that $A \in \mathcal{A}$ implies $A^c \in \mathcal{A}$ and $\{A_i\}_{i=1}^n \subset \mathcal{A}$ then $\cup_{i=1}^n A_i \in \mathcal{A}$.*

**Definition 2.** *($\sigma$-algebra) $\mathcal{A}$ is a $\sigma$-algebra if it is an algebra and for every countable collection of sets $\{A_i\}_{i=1}^\infty \subset \mathcal{A}$ then $\cup_{i=1}^\infty A_i \in \mathcal{A}$.*

Notice that algebras might contain some of their countable unions, but don't have to contain them all. On the other hand, a $\sigma$-algebra is always an algebra, from definition.

If $\mathcal{A}$ is an algebra ($\sigma$-algebra) consider $\cap A_i = (\cup A_i^c)^c$. Then, algebras ($\sigma$-algebras) are closed under finite (countable) intersections. Any algebra must contain the empty set and X, since $A \cap A^c = \phi$ and $A \cup A^c = X$. It is also true that $A \setminus B = A \cap B^c$, so $\sigma$-algebras are closed under set diference.

A $\sigma$-algebra could be a set too big to work with, so we might be interested in smaller sets, here we introduce collections of subsets of X that generate the $\sigma$-algebra.

**Definition 3.** *Let $\mathcal{E} \subset \mathcal{P}(X)$ then the smallest $\sigma$-algebra containing $\mathcal{E}$ is called the $\sigma$-algebra generated by $\mathcal{E}$, denoted by $\mathcal{M}(\mathcal{E})$*

It is easy to see that $\mathcal{M}(\mathcal{E})$ is the intersection of all the $\sigma$-algebras containing $\mathcal{E}$, provided that the intersection is a $\sigma$-algebra.

**Lemma 1.** *If $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$ then $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$*

*Proof.* As $\mathcal{E} \subset \mathcal{M}(\mathcal{F})$ then as $\mathcal{M}(\mathcal{F})$ is closed under countable unions and complements, we must have that $\mathcal{M}(\mathcal{E}) \subset \mathcal{M}(\mathcal{F})$. □

**Definition 4.** *Let X be a topological space, then the collection of all open sets of X is a $\sigma$-algebra, known as the Borel $\sigma$-algebra, $\mathcal{B}_X$. For the real metric space we denote it by $\mathcal{B}_\mathbb{R}$*

**Proposition 1.** *$\mathcal{B}_\mathbb{R}$ is generated by each of the following:*

(1) *the open intervals: $\mathcal{E}_1 = \{(a, b) : a < b\}$*
(2) *the closed intervals: $\mathcal{E}_2 = \{[a, b] : a < b\}$*

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(3) *the half-open intervals:* $\mathcal{E}_3 = \{[a,b) : a < b\}$ *and* $\mathcal{E}_4 = \{(a,b] : a < b\}$
(4) *the open rays:* $\mathcal{E}_5 = \{(a,\infty) : a \in \mathbb{R}\}$ *and* $\mathcal{E}_6 = \{(-\infty,b) : b \in R\}$
(5) *the closed rays:* $\mathcal{E}_7 = \{[a,\infty) : a \in \mathbb{R}\}$ *and* $\mathcal{E}_8 = \{(-\infty,b] : b \in R\}$

*Proof.* All the collections except for $j = 3,4$ are open or closed sets, so $\mathcal{E}_j \subset \mathcal{B}_\mathbb{R}$ and by Lemma (1) we have that $\mathcal{M}(\mathcal{E}_j) \subset \mathcal{B}_\mathbb{R}$. Now for $j = 3,4$ we have that the collections are $G_\delta$, i.e. $(a,b] = \cap_1^\infty (a, b+1/n)$, so then also contained in the Borel $\sigma$-algebra.

As any open set in $\mathbb{R}$ is the union of open intervals, then we have that $\mathcal{B}_\mathbb{R} \subset \mathcal{M}(\mathcal{E}_1)$. We just have to prove now that every $\mathcal{M}(\mathcal{E}_1) \subset \mathcal{M}(\mathcal{E}_j)$ and by Lemma (1), the result will follow. To do this we will write the open intervals as unions or intersectios of each of the families:

(1) $(a,b) = \cup_1^\infty [a+1/n, b-1/n]$
(2) $(a,b) = \cup_1^\infty (a, b-1/n]$ and $(a,b) = \cup_1^\infty [a+1/n, b)$
(3) $(a,b) = (a,\infty) \cap \cup_1^\infty (b+1/n, \infty)$ and $(a,b) = (-\infty,b) \cap \cup_1^\infty (-\infty, a-1/n)$
(4) $(a,b) = \cup_1^\infty [a+1/n, \infty) \cap [b,\infty)$ and $(a,b) = \cup_1^\infty (-\infty, b+1/n] \cap (-\infty, a]$

This completes the proof. □

We have defined a $\sigma$-algebra for the real line, that will be the device we will use to define a measure on the real line, now we should consider how to define a $\sigma$-algebra for a product space, expecting to use this to produce measures in $\mathbb{R}^n$.

**Definition 5.** *(Product $\sigma$-algebra) Let $\{X_\alpha\}_{\alpha\in A}$ be an indexed collection of sets, equipped with $\mathcal{M}_\alpha$ as a $\sigma$-algebra for every $\alpha \in A$. Let $X = \prod_{\alpha\in A} X_\alpha$, then the product $\sigma$-algebra on $X$ is the $\sigma$-algebra generated by:*

$$\otimes_{\alpha\in A}\mathcal{M}_\alpha = \left\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\right\}$$

*Where $\pi_\alpha$ is the projection mapping to the $\alpha$ coordinate.*

**Proposition 2.** *If $A$ is countable then $\otimes_{\alpha\in A}\mathcal{M}_\alpha$ is the $\sigma$-algebra generated by $\mathcal{M} = \left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{M}_\alpha\right\}$*

*Proof.* For the first inclusion, we note that $\pi_\alpha^{-1}(E_\alpha) = \prod_{\beta\in A} E_\beta$ where $E_\beta = X_\beta$ for $\beta \neq \alpha$, then by lemma (1) $\otimes_{\alpha\in A}\mathcal{M}_\alpha \subset \mathcal{M}$.

Now the second inclusion follows from $\prod_{\alpha\in A} E_\alpha = \cap_{\alpha\in A}\pi_\alpha^{-1}(E_\alpha)$, so as the $\sigma$-algebra is closed under countable intersections, we have that $\mathcal{M} \subset \otimes_{\alpha\in A}\mathcal{M}_\alpha$. □

**Proposition 3.** *Suppose that $\mathcal{M}_\alpha$ is generated by $\mathcal{E}_\alpha$ for $\alpha \in A$, then $\otimes_{\alpha\in A}\mathcal{M}_\alpha$ is generated by $\mathcal{F}_1 = \left\{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha, \alpha \in A\right\}$.*

*Proof.* By definition $\mathcal{M}(\mathcal{F}_1) \subset \otimes_{\alpha\in A}\mathcal{M}_\alpha$. So, we are left with the other inclusion. Consider that $\left\{E \in X_\alpha : \pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F}_1)\right\}$ is a $\sigma$-algebra on $X_\alpha$ that contains $\mathcal{E}_\alpha$, and by Lemma (1), it contains $\mathcal{M}_\alpha$. This means that $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{M}(\mathcal{F}_1)$, for all $E \in \mathcal{M}_\alpha$ then, by Lemma (1) we conclude that $\otimes_{\alpha\in A}\mathcal{M}_\alpha \subset \mathcal{M}(\mathcal{F}_1)$, as desired. □

**Corollary 1.** *If $A$ is countable and $X_\alpha \in \mathcal{E}_\alpha$ for all $\alpha \in A$, then $\otimes_{\alpha\in A}\mathcal{M}_\alpha$ is generated by $\mathcal{F}_2 = \left\{\prod_{\alpha\in A} E_\alpha : E_\alpha \in \mathcal{E}_\alpha\right\}$.*

*Proof.* This result follows from Proposition (3) and Proposition (2) □

**Proposition 4.** *Let $X_1, ..., X_n$ be metric spaces and let $X = \prod_1^n X_j$, equipped with the product metric. Then $\otimes_1^n \mathcal{B}_{X_j} \subset \mathcal{B}_X$. If the $X_j$'s are separable, then $\otimes_1^n \mathcal{B}_{X_j} = \mathcal{B}_X$*

*Proof.* By Proposition (3), we know that $\otimes_1^n \mathcal{B}_{X_j}$ is generated by $\pi_\alpha^{-1}(U_j)$ where $U_j$ are open sets in $X_j$, then $\otimes_1^n \mathcal{B}_{X_j} \subset \mathcal{B}_X$.

Now consider that the $X_j$'s are separable, then take $C_j$ to be a countable dense set in $X_j$ and let $\mathcal{E}_j$ be the set of open balls with rational radii and center in $C_j$, clearly $\mathcal{E}_j$ is countable for every $1 \leq j \leq n$. As $X_j$ is separable, then every element $\pi_j(E)$ where $E \in \mathcal{B}_X$ is the countable union of elements of $\mathcal{E}_j$, then in $\mathcal{B}_{X_j}$, now this is true for every $j$, then $\mathcal{E}_j$ generate $\mathcal{B}_{X_j}$, by Proposition (3), we conclude that $\otimes_1^n \mathcal{B}_{X_j} = \mathcal{B}_X$. □

**Definition 6.** *Let $X$ be a set, then $\mathcal{E}$ is an elementary family of $X$ if $\mathcal{E}$ is a collection of subsets of $X$, such that:*

1. *$\phi \in \mathcal{E}$,*
2. *if $E, F \in \mathcal{E}$ then $E \cap F \in \mathcal{E}$,*
3. *if $E \in \mathcal{E}$ then $E^c$ is a finite disjoint union of members of $\mathcal{E}$.*

**Proposition 5.** *If $\mathcal{E}$ is an elementary family, then the collection of finite disjoint unions of members of $\mathcal{E}$, namely $\mathcal{A}$ is an algebra.*

*Proof.* If $A, B \in \mathcal{E}$ then $B^c = \cup^n C_j$ with $C_j \in \mathcal{E}$ and disjoint. Consider $A \setminus B = A \cap B^c = A \cap (\cup^n C_j) = \cup^n (A \cap C_j)$ where $A \cap C_j$ are disjoint and contained in $\mathcal{E}$ by definition of elementary family. Now $A \cup B = (A \setminus B) \cup B$, thus the finite union of disjoint elements of $\mathcal{E}$, therefore $A \cup B \in \mathcal{A}$, by induction we obtain the finite case.

To prove that $\mathcal{A}$ is closed under complements. Consider $\{A_j\}_1^n \subset \mathcal{A}$, then $A_j^c = \cup^m B_{i,j}$ where the $B_{i,j} \in \mathcal{E}$ and disjoint. Then consider

$$\left(\bigcup_{j=1}^n A_j\right)^c = \left(\bigcap_{j=1}^n A_j^c\right) = \left(\bigcap_{j=1}^n \left(\bigcup_{i=1}^m B_{i,j}\right)\right) = \left(\bigcup_{i=1}^m \left(\bigcap_{j=1}^n B_{i,j}\right)\right)$$

which is in $\mathcal{A}$. □

## 2. Problems

**Problem 1.** *A family of sets $\mathcal{R} \subset \mathcal{P}(X)$ is called a* ***ring*** *if it is closed under finite unions and differences. A ring that is closed under countable unions is called a $\sigma$-ring.*

1. *Rings ($\sigma$-rings) are closed under finite (countable) intersections*
2. *If $\mathcal{R}$ is a ring ($\sigma$-ring), then $\mathcal{R}$ is an algebra ($\sigma$-algebra) if, and only if, $X \in \mathcal{R}$*
3. *If $\mathcal{R}$ is a $\sigma$-ring, then $\mathcal{A} = \{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a $\sigma$-algebra.*
4. *If $\mathcal{R}$ is a $\sigma$-ring, then $\mathcal{A} = \{E \subset X : E \cap F \in \mathcal{R} \ \forall\ F \in \mathcal{R}\}$ is a $\sigma$-algebra.*

*Proof.*
1. $A \cap B = A - (A - B)$ and $\cap_{i=1}^\infty A_1 = A_1 - \cup_{j=2}^\infty (A_1 - A_j)$
2. If $\mathcal{R}$ is an algebra, then $X \in \mathcal{R}$. Now, suppose that $\mathcal{R}$ is a ring such that $X \in \mathcal{R}$, then $A^c = X - A$, thus $\mathcal{R}$ is closed under complements, and is an algebra ($\sigma$-algebra).
3. Let $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$ since the definition of $\mathcal{A}$ is symetric, so $\mathcal{A}$ is closed under complements. Let $A, B \in \mathcal{A}$, if $A, B \in \mathcal{R}$ then $A \cup B \in \mathcal{R}$ by definition of a ring. If $A^c, B^c \in \mathcal{R}$ then, $(A \cup B)^c = (A^c \cap B^c) \in \mathcal{R}$ since, it is closed under countable intersections. The last possible situation is $A \in \mathcal{R}$ and $B^c \in \mathcal{R}$, then $(A \cup B) = (A - B^c) \cup (A \cap B^c) \in \mathcal{A}$ since $\mathbb{R}$ is closed

under differences and countable intersections. This yields by induction the finite case.

(4) Let $\{E_i\} \subset \mathcal{A}$ then $\cup E_i = \cup((E_i \cap F) \cup (E_i - F)) \in \mathcal{A}$ since $\mathcal{R}$ is closed under intersections and under differences. Now, let $E \in \mathcal{A}$ and $F = (F \cap E) \cup (F \cap E^c)$, then every member must be in $\mathcal{R}$, then $E^c \in \mathcal{A}$.

□

**Problem 2.** *Let $\mathcal{M}$ be an infinite $\sigma$-algebra.*

1. *$\mathcal{M}$ contains an infinite sequence of disjoint sets.*
2. *$card(\mathcal{M}) \geq \aleph_1$.*

*Proof.* Let $\{E_i\}$ be a countable collection of sets in $\mathcal{M}$ then, let $F_1 = E_1$, $E_2 = E_2 - E_1$ and so on $F_n = E_n - \cup_{i=1}^{n-1} E_i$. This is a countable sequence of disjoint sets of $\mathcal{M}$.

Then, take any subset $\Theta \subset \mathbb{N}$, then $\cup_{\Theta} F_i$ is an element of $\mathcal{M}$ so $card(\mathcal{M}) \geq card(\mathcal{P}(\mathbb{N})) \geq \aleph_1$. □

**Problem 3.** *An algebra $\mathcal{A}$ is a $\sigma$-algebra iff $\mathcal{A}$ is closed under increasing unions.*

*Proof.* Let $\{E_i\}$ be a countable collection of increasing sets in $\mathcal{A}$, i.e. $E_1 \subset E_2 \subset ...$ then $\cup_{i=1}^{\infty} E_i$ is an element of $\mathcal{A}$, in particular any finite union is an element of $\mathcal{A}$.

Now let $\{E_i\}$ be a ANY countable collection of sets in $\mathcal{A}$, then, make $F_1 = E_1$, and $F_2 = E_2 \cup E_1$, then $F_1 \cup F_2 = E_1 \cup E_2$, but now $F_1 \subset F_2$, in this fashion, we can construct our increasing collection of sets $\{F_i\}$, such that $\cup_{i=1}^{\infty} F_i$ exists in $\mathcal{A}$ and furthermore,$\cup_{i=1}^{\infty} F_i = \cup_{i=1}^{\infty} E_i$, thus, any countable union is contained in $\mathcal{A}$, then $\mathcal{A}$ is a $\sigma$-algebra.

On the other hand, if $\mathcal{A}$ is a $\sigma$-algebra, then is closed under any countable union, this includes the increasing collection of sets. □

**Problem 4.** *If $\mathcal{M}$ is the $\sigma$-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the $\sigma$-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.*

*Proof.* As $\mathcal{F} \subset \mathcal{E}$ then $\mathcal{F} \subset \mathcal{M}(\mathcal{E})$ and by Lemma (1), $\mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{E})$. So if we prove that $\mathcal{M}(\mathcal{F})$ is a $\sigma$-algebra, then necessarily $\mathcal{M}(\mathcal{F}) \supset \mathcal{M}(\mathcal{E})$ because, by definition $\mathcal{E}$ is the smallest $\sigma$-algebra generated by $\mathcal{E}$.

So, let $\{A_i\}$ be a countable collection of subsets of $\mathcal{M}(\mathcal{F})$, then $\cup_{i=1}^{\infty} A_i$ is a countable union of elements of $\mathcal{E}$ since the countable union of countable sets is countable, so $\mathcal{M}_F$ is closed under countable unions. Each $\mathcal{M}(\mathcal{F})$ is closed under complements, so $\mathcal{M}(\mathcal{F})$ is a $\sigma$-algebra, thus $\mathcal{M}(\mathcal{F}) = \mathcal{M}(\mathcal{E})$. □

## 3. Measures

We have spent the last section constructing special collection of sets, denominated $\sigma$-algebras. These objects were created with an application in mind, that is, that they would serve as the domain of specific set functions, so to say, measures.

**Definition 7.** *(Measure) Let $X$ be a set and $\mathcal{M}$ a $\sigma$-algebra on $X$. Then a measure is a set function $\mu : \mathcal{M} \to [0, \infty]$ such that:*

1. $\mu(\phi) = 0$
2. *For any sequence $\{A_n\}_1^{\infty}$ of disjoint sets in $\mathcal{M}$, then $\mu(\cup_1^{\infty} A_n) = \sum_1^{\infty} \mu(A_n)$. (Additivity)*

Suppose that there is a set function $\mu : \mathcal{M} \to [0, \infty]$ such that $\mu$ satisfies (1) but, (2) is true only for finite disjoint sets, then we call $\mu$ a **finitely additive measure**.

It is easy to see that a measure is always a finitely additive measure, considering $A_n = \phi$ for every $n > N$, but the converse is not always true.

**Definition 8.** *Let $X$ be a set equipped with a $\sigma$-algebra $\mathcal{M}$, we call $(X, \mathcal{M})$ a measurable space. If we provide $(X, \mathcal{M})$ with a measure $\mu$ we say that $(X, \mathcal{M}, \mu)$ is a measure space.*

Sometimes we will refer to $\mathcal{M}$ as a measure space, when X and $\mu$ are understood.

**Definition 9.** *(finite, $\sigma$-finite, and semifinite measures) Let $(X, \mathcal{M}, \mu)$ be a measure space.*

1. *$\mu$ is finite, if $\mu$ is such that $\mu(E) < \infty$ for every $E \in \mathcal{M}$.*
2. *$\mu$ is $\sigma$-finite, if $X = \cup_1^\infty E_j$ for a disjoint sequence of sets $\{E_j\}$ in $\mathcal{M}$ and $\mu(E_j) < \infty$ for every $1 \leq j < \infty$. A set*
3. *$\mu$ is semifinite, if for every $E \in \mathcal{M}$ that $\mu(E) = \infty$ there exists $F \in \mathcal{M}$ such that $F \subset E$ and $\mu(F) < \infty$.*

**Lemma 2.** *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

1. *If $E, F \in \mathcal{M}$ and $E \subset F$, then $\mu(E) \leq \mu(F)$. (**Monotonicity**)*
2. *If $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\cup_1^\infty E_j) \leq \sum_1^\infty \mu(E_j)$. (**Subadditivity**)*
3. *If $\{E_j\}_1^\infty \subset \mathcal{M}$ is such that $E_1 \subset E_2 \subset \ldots$, then $\mu(\cup_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$ (**Continuity from below**).*
4. *If $\{E_j\}_1^\infty \subset \mathcal{M}$ is such that $E_1 \supset E_2 \supset \ldots$ and $\mu(E_1) < \infty$, then $\mu(\cap_1^\infty E_j) = \lim_{j\to\infty} \mu(E_j)$ (**Continuity from above**).*

*Proof.* (1) Since $E \subset F$ then $F = E \cup (F - E)$ which are disjoint, then $\mu(E) + \mu(F - E) = \mu(F)$, as $\mu(F - E) \geq 0$, then $\mu(E) \leq \mu(F)$.

(2) Let $\{E_j\}_1^\infty \subset \mathcal{M}$, then $F_1 = E_1$ and $F_n = E_n - (\cup_1^{n-1} E_j)$. We have that $\cup_1^n E_j = \cup_1^n F_j$ for all $n$, and the $F_j$'s are disjoint, then by countable additivity of $\mu$ and monotonicity, we have that,

$$\mu(\cup_1^\infty E_j) = \mu(\cup_1^\infty F_j) = \sum_1^\infty \mu(F_j) \leq \sum_1^\infty \mu(E_j)$$

(3) If we set $E_0 = \phi$ then we can write disjoint $F_n = E_n - E_{n-1}$, so that $\cup_1^n E_n = \cup_1^n F_n$ then,

$$\mu(\cup_1^\infty E_n) = \mu(\cup_1^\infty F_n) = \lim_{n\to\infty} \sum_1^n \mu(F_n)$$

$$= \lim_{n\to\infty} \sum_1^n (\mu(E_n) - \mu(E_{n-1})) = \lim_{n\to\infty} \mu(E_n)$$

(4) Let $F_j = E_1 - E_j$ then $F_1 \subset F_2 \subset \ldots$ and $\mu(E_j) = \mu(E_1) - \mu(F_j)$, since $E_1 = E_j \cup (E_1 - E_j) = E_j \cup F_j$, disjoint, so we now can apply (3), considering that $\cup_1^\infty F_j = E_1 - \cap_1^n E_j$

$$\mu(E_1) = \lim_{n\to\infty} [\mu(\cap_1^n E_j) + \mu(\cup_1^\infty F_j)] = \mu(\cap_1^\infty E_j) + \lim_{n\to\infty} [\mu(E_1) - \mu(E_j)]$$

then,

$$\lim_{n\to\infty} \mu(E_j) = \mu(\cap_1^n E_j)$$

□

**Proposition 6.** *Let $(X,\mathcal{M},\mu)$ be a measure space. If $\mu$ is $\sigma$-finite, then $\mu$ is semifinite.*

*Proof.* Let $A \in \mathcal{M}$ be such that $\mu(A) = \infty$, as $\mu$ is $\sigma$-finite then $X = \cup_1^\infty E_j$ for a disjoint sequence of sets $\{E_j\}$ in $\mathcal{M}$. Then $A = A \cap X = \cup_1^\infty (E_j \cap A)$. Then any of these $(E_j \cap A) \subset A$, and by monotonicity of $\mu$ (Lemma (2)), then $\mu(E_j \cap A) < \mu(E_j) < \infty$. □

**Definition 10.** *Let $(X,\mathcal{M},\mu)$ be a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$, then $N \in \mathcal{N}$ is a* ***null set*** *or a* ***zero-measure set****. If two objects differ in some property only for some $N \in \mathcal{N}$ they are said to have the same property* ***almost everywhere (a.e.)****.*

**Definition 11.** *Let $(X,\mathcal{M},\mu)$ be a measure space, if for every null set $N$, $F \subset N$ implies $F \in \mathcal{N}$, then $\mu$ is a complete measure.*

It would be natural to think that if $F \subset N$ and $\mu(N) = 0$ by monotonicity we would have that $\mu(F) = 0$ but this, needs not to be true, because $F$ not need to be contained in the $\sigma$-algebra $\mathcal{M}$. So, we now would like to have to be able to complete a measure space, this will be furnished in the next,

**Proposition 7.** *Let $(X,\mathcal{M},\mu)$ be a measure space. Let $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$ and $\bar{\mathcal{M}} = \{M \cup F : M \in \mathcal{M}$ and $F \subset N \in \mathcal{N}\}$. Then,*

1. *$\bar{\mathcal{M}}$ is a $\sigma$-algebra,*
2. *There is a unique extension $\bar{\mu}$ of $\mu$ to a complete measure on $\bar{\mathcal{M}}$.*

*Proof.* (1) As $\mathcal{M}$ and $\mathcal{N}$ are closed under countable unions, so is $\bar{\mathcal{M}}$. We need to prove that $\bar{\mathcal{M}}$ is closed under complements. Take $E \cup F \in \bar{\mathcal{M}}$, where $E \in \mathcal{M}$ and $F \subset N \in \mathcal{N}$, we may consider $E \cap F = \phi$. Then **consider the following relation**: $E \cup F = (E \cup N) \cap (F \cup N^c)$, then $(E \cup F)^c = (E \cup N)^c \cup (N - F)$ then we notice that $(E \cup N)^c \in \mathcal{M}$ and $(N - F) \subset N$ then $(E \cup F)^c \in \bar{\mathcal{M}}$, thus $\bar{\mathcal{M}}$ is a $\sigma$-algebra.

(2) We need $\bar{\mu}$ restricted to $\mathcal{M}$ to be $\mu$, so to say $\bar{\mu}|_{\mathcal{M}} = \mu$ (i.e. $\bar{\mu}(E) = \mu(E)$ for $E \in \mathcal{M}$). Let $\bar{\mu}$ be defined as $\bar{\mu}(E \cup F) = \mu(E)$. We must prove that $\bar{\mu}$ is well defined. So take $E_1 \cup F_1 \in \bar{\mathcal{M}}$ and $E_2 \cup F_2 \in \bar{\mathcal{M}}$ so that $E_1 \cup F_1 = E_2 \cup F_2$ for $F_j \subset N_j \in \mathcal{N}$, then $E_1 \subset E_2 \cup F_2$ then by monotonicity $\bar{\mu}(E_1) \leq \bar{\mu}(E_2 \cup F_2) = \bar{\mu}(E_2)$ similarly $\bar{\mu}(E_2) \leq \bar{\mu}(E_1)$ and $\bar{\mu}$ is well defined.

Now we want to prove that $\bar{\mu}$ is a measure. The measure of the empty set follows from assumption. Then, take a countable collection of disjoint sets $\{E_i \cup F_i\}_1^\infty \subset \bar{\mathcal{M}}$, consider

$$\bar{\mu}(\cup_1^\infty (E_i \cup F_i)) = \bar{\mu}((\cup_1^\infty E_i) \cup (\cup_1^\infty F_i))$$

$$= \mu(\cup_1^\infty E_i) = \sum_1^\infty \mu(E_i) = \sum_1^\infty \bar{\mu}(E_i \cup F_i)$$

Since, $(\cup_1^\infty E_i \in \mathcal{M}$ and $\cup_1^\infty F_i \subset \cup_1^\infty N_j \subset \mathcal{N}$ so, $\bar{\mu}$ is a measure on $\bar{\mathcal{M}}$ and is the extension of $\mu$ to $\bar{\mathcal{M}}$.

We now have to prove that $\bar{\mu}$ is complete in $\bar{\mathcal{M}}$. Take $E \cup F \in \bar{\mathcal{M}}$, so that $0 = \bar{\mu}(E \cup F) = \mu(E)$, then $E \in \mathcal{N}$, so if $G \subset E \cup F \subset N \in \mathcal{N}$ then $M \cup G \in \bar{M}$ for every $M \in \mathcal{M}$, particularly $G = G \cup \phi \in \bar{\mathcal{M}}$, so $\bar{\mu}$ is complete.

We are left now only with the uniqueness, suppose that $\hat{\mu}$ is another completion of $\mu$ on $\bar{\mathcal{M}}$ but then take $E \cup F \in \bar{\mathcal{M}}$ then $\hat{\mu}(E \cup F) = \mu(E) = \bar{\mu}(E \cup F)$ and so $\hat{\mu} = \bar{\mu}$. This completes the proof.

□

## 4. Problems

**Problem 5.** *Given a measure space $(X, \mathcal{M}, \mu)$, let $E \in \mathcal{M}$ then define, $\mu_E = \mu(A \cap E)$ for every $A \in \mathcal{M}$, then $\mu_E$ is a measure.*

*Proof.* $\mu_E(\phi) = \mu(E \cap \phi) = \mu(\phi) = 0$

Let $\{A_i\}$ be a countable sequence in $\mathcal{M}$ then $\mu_E(\cup A_i) = \mu(E \cap (\cup_i A_i)) = \mu(\cup_i (E \cap A_i)) = \sum \mu(E \cap A_i) = \sum \mu_E A_i$, this is true, because as the $A_i$ are disjoint, then the $E \cap A_i$ are disjoint. So $\mu_E$ is a measure. □

**Problem 6.** *If $\mu_1, ..., \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, ..., a_n \in [0, \infty]$, then $\mu = \sum_1^\infty a_j \mu_j$ is a measure*

*Proof.* $\mu(\phi) = \sum_1^\infty a_j \mu_j(\phi) = 0$. Let $\{A_j\}_1^\infty$ be a collection of disjoint sets in $\mathcal{M}$ then $\mu(\cup_{k=1}^\infty A_k) = \sum_{j=1}^\infty a_j \mu_j(\cup_{k=1}^\infty A_k)$, but each $\mu_j$ is a measure, thus countably additive, then $= \sum_{j=1}^\infty a_j \sum_{k=1}^\infty \mu_j(A_k) = \sum_{k=1}^\infty \sum_{j=1}^\infty a_j \mu_j(A_k) = \sum_{k=1}^\infty \mu(A_k)$, then $\mu$ is a measure. □

**Problem 7.** *If $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_i\} \subset \mathcal{M}$. Then,*

1. *$\mu(\liminf E_i) \leq \liminf \mu(E_i)$ and*
2. *$\mu(\limsup E_i) \geq \limsup \mu(E_i)$ provided that $\mu(\cup E_i) < \infty$.*

*Proof.*

1. $\liminf E_i = \cup_{k=1}^\infty (\cap_{i \geq k} E_i)$, consider $F_j = \cap_{i \geq j} E_i$ then $F_1 \subset F_2 \subset ...$, then by continuity from below we have that $\mu(\cup_1^\infty F_j) = \mu(\liminf E_i) = \lim_{n \to \infty} \mu(F_n) = \lim_{n \to \infty} \mu(\cap_{i \geq n} E_i) = \liminf \mu(E_i)$.
2. In the same fashion as (1), just consider that in order to apply continuity from above, we need, to have that $\mu(E_j) < \infty$ for a finite natural $j$. So we may assume $j = 1$.

□

**Problem 8.** *If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then*

1. *$\mu(E) + \mu(F) - \mu(E \cap F) = \mu(E \cup F)$.*
2. *If $\mu(E \Delta F) = 0$ then $\mu(E) = \mu(F)$.*
3. *Let $E \sim F$ if $\mu(E \Delta F) = 0$, then $\sim$ defines an equivalence relation.*
4. *For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \Delta F)$. Then $\rho$ defines a metric on the space $\mathcal{M}/\sim$ of equivalence classes.*

*Proof.*

1. Since $E \cup F = (E \cap F) \cup (E \cap F^c) \cup (F \cap E^c)$ and all are disjoint sets, then $\mu(E \cup F) = \mu(E \cap F) + \mu(E \cap F^c) + \mu(F \cap E^c)$. Also consider $E = (E \cap F^c) \cup (E \cap F)$ and $F = (F \cap E^c) \cup (F \cap E)$, then $\mu(E) + \mu(F) - 2\mu(E \cap F) = \mu(E \cap F^c) + \mu(F \cap E^c)$, substituting this result in the first equation, we obtain $\mu(E) + \mu(F) - \mu(E \cap F) = \mu(E \cup F)$.
2. Given that $E \Delta F = (E \cap F^c) \cup (F \cap E^c)$ then $\mu(E \cap F^c) + \mu(F \cap E^c) = \mu(E \Delta F) = 0$. Then, $\mu(E \cap F^c) = \mu(F \cap E^c) = 0$ so, $\mu(E) - \mu(F) = \mu(E \cap F^c) - \mu(F \cap E^c) = 0$ and $\mu(E) = \mu(F)$.
3. By previous item, it is trivial.

(4) All axioms for a metric space are trivial except for the triangle inequality. So, let $E, F, G \in \mathcal{M}$ then $E\Delta G = (E \cap G^c) \cup (G \cap E^c)$ and $F\Delta E = (E \cap F^c) \cup (F \cap E^c)$, then $\rho(E, G) + \rho(G, F) = \mu(E \cap G^c) + \mu(G \cap E^c) + \mu(G \cap F^c) + \mu(F \cap G^c) \geq \mu((E \cap G^c) \cup (G \cap E^c) \cup (G \cap F^c) \cup (F \cap G^c)) = \mu(((E \cup F) \cap G^c) \cup (G \cap (E^c \cap F^c))) = \mu((E \cup F)\Delta G) \geq \mu(E\Delta F) = \rho(E, F)$. The last inequality follows from $E\Delta F \subset (E\Delta G) \cup (F\Delta G)$.

□

**Problem 9.** *If $\mu$ is a semifinite measure and $\mu(E) = \infty$ then for any $C > 0$ there is a set $F \subset E$ such that $C < \mu(F) < \infty$.*

*Proof.* By contradiction, suppose that for $C > 0$ there is no such $F \subset E$ then $\mu(F) < C$ and also $\mu(F^c) < C$, by assumption, then $\mu(E) < \mu(E \cap F) + \mu(E \cap F^c) \leq \mu(F) + \mu(F^c) < 2C$ contradicting $\mu(E) = \infty$, so such $F$ must exist. □

**Problem 10.** *A finitely additivie measure $\mu$ is a measure if, and only if, it is continuous from below. (The result holds for continuous from above if $\mu$ if finite).*

*Proof.* $\mu(\phi) = 0$ since $\mu$ is a finitely additive measure. Now take $\{A_j\}_1^\infty$ a collection of disjoint measurable sets, then let $B_n = \cup_1^n A_j$, then we have that $B_1 \subset B_2 \subset ...$ then we have that, $\mu(\cup_1^\infty A_j) = \mu(\cup_1^\infty B_j) = \lim_{n\to\infty} \mu(B_n) = \lim \sum_1^n \mu(A_n) = \sum_1^\infty \mu(A_n)$, then $\mu$ is a measure.

□

**Problem 11.** *Given a measure space $(X, \mathcal{M}, \mu)$, a set $E \subset X$ is called locally measurable if $E \cap A \in \mathcal{M}$ for all $A \in \mathcal{M}$ such that $\mu(A) < \infty$. Let $\tilde{\mathcal{M}}$ be the collection of all locally measurable sets. Clearly $\mathcal{M} \subset \tilde{\mathcal{M}}$, if $\mathcal{M} = \tilde{\mathcal{M}}$, then $\mu$ is called saturated.*

(1) *If $\mu$ is $\sigma$-finite, then $\mu$ is saturated*
(2) *$\tilde{\mathcal{M}}$ is a $\sigma$-algebra.*
(3) *Define $\tilde{\mu}$ on $\tilde{\mathcal{M}}$ by $\tilde{\mu}(E) = \mu(E)$ for $E \in \mathcal{M}$ and $\tilde{\mu}(E) = \infty$ otherwhise. Then $\tilde{\mu}$ is saturated in $\tilde{\mathcal{M}}$.*
(4) *If $\mu$ is closed, so is $\tilde{\mu}$*

*Proof.* (1) Let $\mu$ be $\sigma$-finite and take any $E \subset \tilde{\mathcal{M}}$, then $E \subset \cup E_i$ for some $E_i$'s. As $\mathcal{M}$ is a $\sigma$-algebra any union of such $E_i$ is an element of $\mathcal{M}$ so, $E \cap (\cup E_i) \in \mathcal{M}$, and $\mu(E_i) < \infty$, then $\tilde{\mathcal{M}} \subset \mathcal{M}$.

(2) Let $\{E_i\}$ be a disjoint countable collection of locally measurable sets, then take any $A \in \mathcal{M}$ such that $\mu(A) < \infty$. Then $\mu((\cup E_i) \cap A) = \mu(\cup(E_i \cap A)) = \sum \mu(E_i \cap A)$ since for each $i$, $E_i \cap A \in \mathcal{M}$.

Now, if E is a locally measurable set, A be the same as before, then $E^c \cap A = A \setminus (E \cap A)$ where $A \in \mathcal{M}$ and $E \cap A \in \mathcal{M}$ then $E^c \cap A \in \mathcal{M}$, thus $E^c \in \tilde{\mathcal{M}}$.

(3) It is inmediate that, $\mathcal{M} \subset \tilde{\mathcal{M}}$. Now, take E a locally measurable set with respect to $\tilde{\mu}$, this means that for every $A \in \tilde{\mathcal{M}}$ such that $\tilde{\mu}(A) < \infty$ then $E \cap A \in \tilde{\mathcal{M}}$ actually $E \cap A \in \mathcal{M}$, so $\tilde{\mathcal{M}}$ is already saturated, i.e. we can not attain $\tilde{\mathcal{M}}$ with more locally measurable sets.

(4) Suppose $\mu$ is closed, then for any $A \subset N \in \mathcal{M}$ such that $\mu(N) = 0$ then $\mu(A) = 0$, so then also is $\tilde{\mu}(N) = 0$, but we must have that $A \in \mathcal{M}$, since $A \cap N = A$, then $\tilde{\mu}(A) = 0$, and $\tilde{\mu}$ is complete.

□

## 5. Outer Measures

**Definition 12.** *Let $X$ be a nonempty set, then $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is an outer measure if,*

1. $\mu^*(\phi) = 0$
2. *For $A, B \in \mathcal{P}(X)$ and $A \subset B$ then $\mu^*(A) \leq \mu^*(B)$.*
3. *For $\{A_j\}_1^\infty \subset \mathcal{P}(X)$ then $\mu^*(\cup_1^\infty A_j) \leq \sum_1^\infty \mu^*(A_j)$.*

**Proposition 8.** *Let $\mathcal{E} \subset \mathcal{P}(X)$ and $\rho : \mathcal{E} \to [0, \infty]$, such that $\phi \in \mathcal{E}$, $X \in \mathcal{E}$ and $\rho(\phi) = 0$, for any $A \subset X$, define,*

$$\mu^*(A) = inf \left\{ \sum_1^\infty \rho(A_j) : A_j \in \mathcal{E} \text{ and } A \subset \cup_1^\infty A_j \right\}$$

*then $\mu^*$ is an outer measure.*

*Proof.* First of all $\mu^*$ is well defined, since for every A, there is at least one cover, since $X \in \mathcal{E}$ and $A \subset X$, so at least $A_j = X$ for all $j$ is a covering for E. Now, if $A \subset B$ every covering that covers $B$ certainly covers $A$, so $\mu^*(A) \leq \mu^*(B)$.

To prove subadditivity we may take $\{A_j\}_{j=1}^\infty \subset \mathcal{P}(X)$. By definition of inf, given $\epsilon > 0$, there is a covering $\{E_{jk}\} \subset \mathcal{E}$ such that $A_j \subset \cup_k E_{jk}$ such that $\sum_j \rho(E_{jk}) \leq \mu^*(A_j) + \epsilon/2^j$, then consider that $A = \cup_1^\infty A_j \subset \cup_{j,k}^\infty E_{jk}$, then

$$\mu^*(\cup_1^\infty A_j) \leq \sum_{j,k}^\infty \rho(E_{jk}) \leq \sum_1^\infty \left[\mu^*(A_j) + \epsilon/2^j\right] = \sum_1^\infty \mu^*(A_j) + \epsilon$$

since $\epsilon > 0$ was arbitrary, we have that $\mu^*(\cup_1^\infty A_j) \leq \sum_1^\infty \mu^*(A_j)$ □

**Definition 13.** *Let $E \subset X$ and $\mu^*$ an outer measure, then $E$ is a $\mu^*$-measurable set if for every $A \subset X$ then, $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$.*

**Theorem 1.** *(Carathedory's Theorem) Let $\mu^*$ be an outer measure on the set $X$.*

1. *$\mathcal{M} = \{E \subset X : E \text{ is } \mu^* - measurable\}$ is a $\sigma$-algebra,*
2. *$\mu^*|_\mathcal{M}$ is a complete measure.*

*Proof.* First we may prove that $\mathcal{M}$ is a $\sigma$-algebra. We notice that $\mathcal{M}$ is closed with respect to complements since the definition of $\mu^*$-measurable sets is symetric. Now, we will prove that $\mathcal{M}$ is an algebra, i.e. $\mathcal{M}$ is closed under finite unions.

Let $A, B \in \mathcal{M}$ then take $E \subset X$, and then, $\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c)$, and applying that B is $\mu^*$-measurable, we obtain that,

$$\mu^*(E) = \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c)$$

Then notice that $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$ all disjoint, so by subadditivity of $\mu^*$ we may write, $\mu^*(A \cup B) \leq \mu^*(A \cap B) + \mu^*(A \cap B^c) + \mu^*(A^c \cap B)$, so,

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$$

by subadditivity we have the other inequality, and thus, $\mu^*(E) = \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c)$, then $A \cup B \in \mathcal{M}$ and simple induction, yields the result for any finite union. So, right now we have proved that $\mathcal{M}$ is an algebra.

We may prove now the result for countable unions. So, let $\{A_j\}_1^\infty$ be a collection of disjoint sets in $\mathcal{M}$, $B_n = \cup_1^n A_j$, and $B = \cup_1^\infty A_j$, then take any $E \subset X$,

$$\mu^*(E \cap B_n) = \mu^*(E \cap B_n \cap A_n) + \mu^*(E \cap B_n \cap A_n^c)$$

$$= \mu^*(E \cap A_n) + \mu^*(E \cap (\cup_1^{n-1} A_j)) = \sum_1^n \mu^*(E \cap A_j)$$

Now we may consider that $B_n \subset B_{n+1} \subset B$ then $B^c \subset B_{n+1}^c \subset B_n^c$, so by monotonicity $\mu^*(E \cap B^c) \leq \mu^*(E \cap B_n^c)$,

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \cap B_n^c) \geq \sum_1^n \mu^*(E \cap A_j) + \mu^*(E \cap B^c)$$

Then, this is true for any n, so we may take the limit and then apply subadditivity of $\mu^*$ to obtain,

$$\mu^*(E) \geq \sum_1^\infty \mu^*(E \cap A_j) + \mu^*(E \cap B^c) \geq \mu^*(E \cap B) + \mu^*(E \cap B^c) \geq \mu^*(E)$$

So we have that $B = \cup_1^\infty A_j \in \mathcal{M}$ and then $\mathcal{M}$ is a $\sigma$-algebra, furthermore, $\mu^*$ is countably additive, letting $E = B$ in the last equation, so $\mu^*$ is a measure in $\mathcal{M}$ we just need now to prove completeness.

Let $A \subset X$ be such that $\mu^*(A) = 0$ then take any $E \subset X$ and then by subadditivity $\mu^*(E) \leq \mu^*(A \cap E) + \mu^*(E \cap A^c) = \mu^*(E \cap A^c) \leq \mu^*(E)$ then $A \in M$ so $\mu^*|_{\mathcal{M}}$ is complete. □

**Definition 14.** *Let $\mathcal{A} \in \mathcal{P}(X)$ be an algebra, a function $\rho : \mathcal{A} \to [0, \infty]$ is called a premeasure if,*

1. $\rho(\phi) = 0$
2. *If $\{A_j\}_1^\infty$ is a sequence of disjoint elements in $\mathcal{A}$ such that $\cup_1^\infty A_j \in \mathcal{A}$, then $\rho(\cup_1^\infty A_j) = \sum_1^\infty \rho(A_j)$*

A premeasure $\rho$ induces an outer measure $\mu^*$ defining, for every $E \subset X$,

$$\mu^*(E) = \inf \left\{ \sum_1^\infty \rho(A_j) : A_j \in \mathcal{A} \text{ and } E \subset \cup_1^\infty A_j \right\}$$

**Proposition 9.** *Let $X$ be a set, $\mathcal{A}$ an algebra on $X$, $\mu_0$ a premeasure on $\mathcal{A}$ and, $\mu^*$ be the outer measure induced by $\mu_0$, then,*

1. $\mu^*|_{\mathcal{A}} = \mu_0$,
2. *every $A \in \mathcal{A}$ is a $\mu^*$-measurable set.*

*Proof.*

1. Let $E \in \mathcal{A}$ then $E \subset \cup_1^\infty A_j$ for a sequence of sets in $\mathcal{A}$. Then let $B_j = E \cap (A_j - \cup_1^{j-1} A_j)$ then $E = \coprod_1^\infty B_j$ and then $\mu_0(E) = \sum_1^\infty B_j \leq \sum_1^\infty A_j$ then $\mu_0(E) \leq \mu^*(E)$ the other inclusion is trivial since $E \subset \cup_1^\infty A_j$.
2. Let $A \in \mathcal{A}$ and $E \subset X$, given $\epsilon > 0$ there is a collection of sets $\{A_j\}$ in $\mathcal{A}$ such that $E \subset \cup A_j$ and $\mu^*(E) + \epsilon \geq \sum_1^\infty \mu_0(A_j) \geq \sum_1^\infty \mu_0(A \cap A_j) + \sum_1^\infty \mu_0(A^c \cap A_j) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$.

□

**Theorem 2.** *Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mu_0$ a premeasure on $\mathcal{A}$, and $\mathcal{M}$ the $\sigma$-algebra generated by $\mathcal{A}$. Let $\mu^*$ be the outer measure induced by $\mu_0$. Then,*

1. *There exists a measure $\mu$ on $\mathcal{M}$ whose restriction to $\mathcal{A}$ is $\mu_0$. Say $\mu^*|_{\mathcal{M}} = \mu$.*
2. *If $\nu$ is another measure that extends $\mu_0$, then $\nu(E) \leq \mu(E)$, for all $E \in \mathcal{M}$, with equality when $\mu(E) < \infty$.*
3. *If $\mu_0$ is $\sigma$-finite, then $\mu$ is the unique extension of $\mu_0$ to a measure on $\mathcal{M}$.*

*Proof.* (1) Let $\mathcal{M}^*$ the $\sigma$-algebra of $\mu^*$-measurable sets. By Proposition (9), we know that $\mathcal{A} \subset \mathcal{M}^*$ and by Lemma (1), we know that $\mathcal{M} \subset \mathcal{M}^*$, so, by Carathedory's theorem, we have that there is a measure $\mu$ so that $\mu^*|_{\mathcal{M}^*} = \mu$ a complete measure in $\mathcal{M}^*$, thus, a measure in $\mathcal{M}$.

(2) If $E \subset X$ and $E \subset \cup A_j$ then $\nu(E) \leq \sum \nu(A_j) = \sum \mu(A_j)$ then $\nu(E) \leq \mu(E)$. (Equality pending)

(3) Suppose $X = \cup A_j$ where $\mu_0(A_j) < \infty$ for every $j$, we may assume that the $A_j$'s are disjoint, then $\mu(E) = \sum \mu(E \cap A_j) = \sum \nu(E \cap A_j) = \nu(E)$, then $\nu = \mu$.

□

## 6. Problems

**Problem 12.** *If $\mu^*$ is an outer measure on $X$ and $\{A_j\}$ is a sequence of disjoint $\mu^*$-measurable sets, then $\mu^*(E \cap (\cup A_j)) = \sum \mu^*(E \cap (\cup A_j))$.*

*Proof.* In the proof of Carathodory theorem, we proved that $\mu^*(E) = \sum_1^\infty \mu^*(E \cap A_j) + \mu^*(E \cap B^c)$ let $B = \cup_1^\infty A_j$ and apply the result to $E \cap (\cup_1^\infty A_j)$, and we obtain the desired result. □

**Problem 13.** *Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$ and $A_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let $\mu_0$ be a premeasure on $\mathcal{A}$ and $\mu^*$ the induced outer measure.*

(1) *For any $E \subset X$ and $\epsilon > 0$ there exists $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$*

(2) *If $\mu^*(E) < \infty$ then $E$ is $\mu^*$-measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.*

(3) *If $\mu_0$ is $\sigma$-finite, the restriction $\mu^*(E) < \infty$ in (2) is superfluous.*

*Proof.* (1) Take $E \subset X$ and $\epsilon > 0$, then let $\{A_i\} \subset \mathcal{A}$ such that $E \subset \cup A_i$, this is possible, since $\mathcal{A}$ is an algebra and $E \subset X$. So $\mu^*(E) = \inf \{\sum \mu_0(A_i) : A_i \in \mathcal{A} \text{ and } E \subset \cup A_i\}$. Then $A = \cup A_i \in \mathcal{A}_\sigma$, and we will have $\mu^*(A) \leq \mu^*(E) + \epsilon$

(2) If $\mu^*(E) < \infty$ then suppose $E$ is $\mu^*$-measurable, then we may take a sequence of $\{B_n\} \subset \mathcal{A}_\sigma$ so that $B_1 \supset B_2 \supset ...$ so that $B = \cap_1^\infty B_n \in \mathcal{A}_{\sigma\delta}$ and $E \subset B$. By previous item, for every $n \in \mathbb{N}$ there is a $B_n \in \mathcal{A}_\sigma$ so that $\mu^*(B_n) \leq \mu^*(E) + 1/n$, by continuity from above, we have that $\mu(B) = \lim \mu(B_n) \leq \lim \mu^*(E) + 1/n = \mu(E)$, then $\mu(B) = \mu(B \cap E) + \mu(B \setminus E) = \mu(E) + \mu(B \setminus E)$, then $\mu(B \setminus E) = 0$.

Suppose that for $E \subset X$ such that $\mu^*(E) < \infty$ there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$, then for any $F \subset X$ consider $F \setminus E \subset (F \setminus B) \cup (B \setminus E)$ then $\mu^*(F \cap E) + \mu^*(F \setminus E) \leq \mu^*(F \cap B) + \mu^*(F \setminus B) + \mu^*(B \setminus E) = \mu^*(F \cap B) + \mu^*(F \setminus B) = \mu^*(F)$. The other inequality follows from subadditivity, then $E$ is measurable.

(3) For proof of the necessary condition in the previous item, we used the fact that $\mu^*(E) < \infty$ to apply continuity from above. Now, if $\mu^*$ is $\sigma$-finite, then every set is a $\sigma$-finite set, so we can let $E = \cup_1^\infty E_j$ where $\mu^*(E_j) < \infty$, then for each $E_j$ we can find a $B_j \in \mathcal{A}_{\sigma\delta}$ that satisfies the condition of the theorem, then $B = \cup_1^\infty B_j$ is such that $\mu^*(B \setminus E) \leq \sum_1^\infty \mu^*(B_j \setminus E_j) = 0$.

□

## 7. Borel Measures

Recall, that $\mathcal{B}_{\mathbb{R}}$ is the $\sigma$-algebra generated by the collection of all open set in $\mathbb{R}$. By Proposition (1), $\mathcal{B}_{\mathbb{R}}$ is generated by open, closed, half-open intervals and open and closed rays. We will be interested in using the half-open intervals $(a, b], (a, \infty), \phi$ as the building blocks for our theory. Then, let $\mathcal{A}$ be the algebra generated by the h-intervals (half-open intervals).

**Proposition 10.** *Let $F : \mathbb{R} \to \mathbb{R}$ be an increasing and right continuous function. If $\{(a_i, b_i]\}_1^n$ are disjoint h-intervals in $\mathbb{R}$, let,*

$$\mu_0 \left(\cup_{j=1}^n (a_j, b_j]\right) = \sum_{j=1}^n [F(b_j) - F(a_j)]$$

*and $\mu_0(\phi) = 0$ then $\mu_0$ is a premeasure on $\mathcal{A}$.*

*Proof.*

- $\mu_0$ **is well defined:** We may write an interval $(a, b]$ in several ways, we may consider a sequence $\{(a_i, b_i]\}_1^n$ so that $(a, b] = \cup_1^n (a_i, b_i]$, since this collection is finite, we may relabel them an reorder in such way that $a = a_1 \leq b_1 = a_1 \leq ... = a_n \leq b_n = b$, then $\sum_{j=1}^n [F(b_j) - F(a_j)] = F(b) - F(a)$. So for a single interval, $\mu_0$ is well defined. Now lets consider two finite collections of intervals $\{I_k\}_1^n$ and $\{J_k\}_1^m$, such that $\cup_1^n I_k = \cup_1^m J_k$, then in the same way we did for a single interval, we may notice that $\sum_{k=1}^n \mu_0(I_k) = \sum_{j,k} \mu_0(I_k \cap J_j) = \sum_{j=1}^n \mu_0(J_j)$, so $\mu_0$ is well defined.
- $\mu_0(\phi) = 0$ by definition.
- **Countable additivity:** Consider $\{I_j\}_1^\infty$, such that $\cup_1^\infty I_j \in \mathcal{A}$, as $\mathcal{A}$ is the algebra generated by the open intervals, $\cup_1^\infty I_j$ can be written as the finite union of disjoint intervals. So, it suffices to show that the this is true for a collection $\{(a_i, b_i]\}_1^\infty$ such that $\cup_1^\infty (a_i, b_i] = (a, b]$ then we know that given $\epsilon > 0$ there is a $\delta > 0$ so that $F(a + \delta) - F(a) < \epsilon$, and $F(b_n + \delta_n) - F(b_n) < \frac{\epsilon}{2^n}$, since F is right continuous.

  Consider an open covering $\{(a_j, b_j + \delta_j)\}$ for $[a + \delta, b]$, this is compact, so there it must have a finite subcovering, then take the covering $\{(a_j, b_j + \delta_j)\}_1^N$ such that $a_j \in ((a_{j-1}, b_{j-1} + \delta_{j-1}))$ and $b_j \in ((a_{j+1}, b_{j+1} + \delta_{j+1}))$. Then,

$$\mu_0(I) \leq F(b) - F(a + \delta) + \epsilon \leq F(b_N + \delta_N) - F(a_1) + \epsilon$$
$$= F(b_N + \delta_N) - F(a_N) + \sum_1^{N-1} [F(a_{j+1}) - F(a_j)] + \epsilon$$
$$\leq F(b_N + \delta_N) - F(a_N) + \sum_1^{N-1} [F(b_j + \delta_j) - F(a_j)] + \epsilon$$
$$= \sum_1^N [F(b_j + \delta_j) - F(a_j)] + \epsilon \leq \sum_1^N \left[F(b_j) - F(a_j) + \frac{\epsilon}{2^j}\right] + \epsilon$$
$$< \sum_1^\infty [F(b_j) - F(a_j)] + 2\epsilon$$

  The other inequality follow from subadditivity, so we have that $\mu_0$ is a premeasure.

□

**Theorem 3.** *If $F : \mathbb{R} \to \mathbb{R}$ is an increasing and right continuous function, there is a unique Borel measure $\mu_F$ on $\mathbb{R}$ such that $\mu_F((a,b]) = F(b) - F(a)$ for all a,b. If G is another function so that $\mu_F = \mu_G$, then $G - F$ is a constant.*

*Conversely, If $\mu$ is a Borel measure on $\mathbb{R}$ that is finite on all bounded Borel sets and define,*

$$F(x) = \begin{cases} \mu((0,x]) & if \quad x > 0 \\ 0 & if \quad x = 0 \\ -\mu((x,0]) & if \quad x < 0 \end{cases}$$

*Then F is increasing and right continuous and $\mu = \mu_F$.*

*Proof.* By Proposition (10), we know that F induces a $\sigma$-finite premeasure on $\mathcal{A}$, taking that $\mathbb{R} = \cup_{-\infty}^{\infty}(n, n+1]$ and $F(n+1) - F(n) < \infty$ for every $n$. It is also clear that $\mu_F = \mu_G$ if, and only if, $F - G$ is a constant. Now, the result follows from Theorem (2), which constructs the extension for the premeasure as an outer measure and the complete measure as the restriction of this outer measure.

Now for the converse, notice that the monotonicity of $\mu$ implies that F is increasing, and the continuity from above and below imply the right continuity of F, and $\mu = \mu_F$ in $\mathcal{A}$, hence on $\mathcal{B}_{\mathbb{R}}$. □

Notice that Theorem (2) not only gives a measure, but a complete measure $\bar{\mu}_F$ on the completion of $\mathcal{B}_{\mathbb{R}}$. This measure will be called the **Lebesgue-Stieltjes measure** associated to F, and the domain of this measure will be denominated by $\mathcal{M}_\mu$.

The next lemma will show that we could have defined our measure in terms of open intervals, and both definitions are equivalent.

**Lemma 3.** *For any $E \in \mathcal{M}_\mu$,*

$$\mu(E) = \inf\left\{ \sum_1^\infty \mu((a_i, b_j)) : E \subset \bigcup_1^\infty (a_i, b_i) \right\}$$